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Finite-capacity GI/G/1 queueing systems with buffer overflows

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Abstract

In this paper GI/G/1 queueing systems with buffer overflows are involved. First, we will give asymptotic expressions for the long-run fraction of rejected customers, in case the buffer capacity K grows large. Furthermore, we give a non-trivial upper and lower bound for this fraction for all K larger than some constant. Finally, we treat a method to speed up the simulation of the fraction of rejected customers.

• FINITE-CAPACITY QUEUEING SYSTEMS • LONG-RUN FRACTION OF REJECTED CUSTOMERS • LARGE DEVIATIONS • MARTINGALES • RANDOM WALKS • IMPORTANCE SAMPLING

1 Introduction

CONSIDER A QUEUEING model with a single server. Let A_n be the epoch at which the n th customer arrives. We assume that the sequence of interarrival times $\{(A_n - A_{n-1}), n \in \mathbb{N}\}$ consists of independent random variables with an identical distribution. Put $A_0 := 0$. Let $A(\cdot)$ be the cumulative distribution function of A_1 , which we assume to be continuous.

Each customer brings along a random amount of work. Let B_n be the work requirement of the n th customer. We assume that $\{B_n, n \in \mathbb{N}\}$ is a family of random variables sampled from a common distribution. Define by $B(\cdot)$ the (continuous) distribution function of B_1 . We also assume the interarrival process $\{(A_n - A_{n-1}), n \in \mathbb{N}\}$ and the work requirement process $\{B_n, n \in \mathbb{N}\}$ to be independent.

The work brought in by the customers is put into a finite-capacity buffer, that is emptied at a rate of $\sigma > 0$ per unit time. The buffer has finite capacity K . The model can be considered as a queueing model of the GI/G/1 type with buffer overflows.

In the sequel we will consider two models. In the first model, a customer bringing along more work than the remaining capacity in the buffer causes an overflow. The excess of work is lost. This model is also called the *partial rejection* model. We define by $\pi_1(K)$ the long-run fraction of customers who cause an overflow.

In the second model, *complete rejection* is involved. A customer is rejected if the sum of the amount of work yet to be done just before the arrival of this customer and his work requirement exceeds K . Let $\pi_2(K)$ be the long-run fraction of customers who are rejected in this model.

In the case of Poisson arrivals, these models are discussed in Tijms [1986, pages 309-318 and 324-331].

It is assumed that $E(B_1)/\sigma < E(A_1 - A_0)$. This assumption guarantees that the overflow probability can be made arbitrarily small by taking the buffer capacity large enough. An important problem is: how to choose K such that the long-run fraction of customers that cause an overflow does not exceed a given (small) value. To cope with this problem, our goals in this paper are:

- we prove, under certain conditions on the distribution of $A_1 - A_0$ and B_1 , that for some positive constants θ, ζ_1 and ζ_2 , we have that $\pi_i(K)$ is asymptotically exponential:

$$\lim_{K \rightarrow \infty} \pi_i(K) e^{\theta K} = \zeta_i,$$

for $i = 1, 2$. I.e., for large K , $\pi_i(K)$ approximately equals $\zeta_i e^{-\theta K}$, where ζ_i is the amplitude (or prefactor) and θ the decay rate. Note that this decay rate is identical for both partial and complete rejection.

- we find, under the same conditions, some non-trivial functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ such that

$$\eta_1(K) \leq \pi_i(K) \leq \eta_2(K),$$

for $i = 1, 2$ and all K larger than some positive constant.

The organization of the paper is as follows. In section 2, we will investigate the long-run fraction of rejected customers applying the technique of Large Deviations and explain what the drawbacks are of the expressions obtained by this approach.

Section 3 focuses on random walks and Martingale theory. We introduce an exponential martingale for random walks. Then we consider the probability that the random walk (starting in $a > 0$) exceeds K before hitting $(-\infty, 0]$. Using Doob's optional sampling theorem, we prove this probability to be asymptotically exponential ($K \rightarrow \infty$). Furthermore, we derive a non-trivial upper and lower bound for it.

In section 4 we modify the random walk model into the queueing model in question. The results from section 4 enable us to prove that the first objective above holds for the probability of a loss cycle (which we will call $\alpha(K)$ throughout). Using simple arguments, we find again an upper and lower bound.

Section 5 deals with the relation between $\pi_i(K)$ and $\alpha(K)$. Since we found the desired already for $\alpha(K)$, we only have to translate the results for $\alpha(K)$ into results for $\pi_i(K)$. We conclude that we found more exact results than could be realised using Large Deviations theory.

In the last section is dealt with the estimation of $\pi_i(K)$ ($i = 1, 2$) by means of simulation, especially for large values of buffer capacity K . It is explained how to speed up these simulations.

2 Results from Large Deviations theory

IN THIS SECTION we will give an expression for $\pi_i(K)$ (where $i = 1, 2$) using *Large Deviations*. Bucklew [1990] can be regarded as an accessible summary of the main results of this theory. Moreover, it explains how to apply it in engineering. We also refer to Ellis [1985] and Dembo and Zeitouni [1993] for good, but a bit more technical, texts on Large Deviations.

In Large Deviations, rare events are involved. In other words, the probability that a certain stochastic process attains improbable values is considered. In the GI/G/1 queueing system we are dealing with, the rare event is a buffer overflow.

We define by $X_K(t)$ the amount of work in the system at time t , where $t \geq 0$. Let C_m denote the m th cycle, i.e., $C_m = [a_m, b_m)$, satisfying

$$X_K(a_m-) = 0, X_K(b_m) = 0 \text{ and } X(t) > 0 \text{ on } [a_m, b_m).$$

We call a cycle a loss cycle if during this cycle at least one customer causes an overflow. Recalling that $\alpha(K)$ denotes the probability of a loss cycle, we note that this probability is the same for both the partial and complete rejection model. As long as in a cycle no customer is rejected, it is clear that (starting in $t = 0$ with an empty system)

$$X_K(t) = \sum_{n=1}^{\infty} \{B_n \times 1_{[0,t]}(A_n)\} - \sigma(t - A_1) \times 1_{[0,t]}(A_1).$$

But then we have just prior to $t = A_n$

$$X_K(A_n-) = \sum_{i=1}^{n-1} (B_i - \sigma(A_{i+1} - A_i)),$$

$n = 2, 3, \dots$. According to Bucklew [1990, pages 56-74] and Dembo and Zeitouni [1993, pages 152-160], we may apply Slow Markov Walk theory, since we are dealing with a sum of independent, identically distributed random variables. The next characterization of the decay rate of $\alpha(K)$ is due to Mogulskii:

$$\lim_{K \rightarrow \infty} \frac{1}{K} \log \alpha(K) = -\theta,$$

where $\xi_i := B_i - \sigma(A_{i+1} - A_i)$ and θ is a non-negative solution of $E(\exp(\theta \xi_1)) = 1$. Equivalently, we find that

$$\alpha(K) = \zeta(K) e^{-\theta K},$$

for some positive function $\zeta(\cdot)$ with $\log \zeta(K) = o(K)$, $K \rightarrow \infty$. Note that $\zeta(K)$ does not converge necessarily to a constant for $K \rightarrow \infty$.

Large Deviations theory does not only provide us an asymptotical expression for the probability of a loss cycle $\alpha(K)$, but we also obtain the trajectory to level K (where K is large) which is in some sense the most probable path: a straight line with slope $E(\xi_1 \exp(\theta \xi_1))$, $n \in \mathbb{N}_0$. It can be concluded that the optimum path to reach overflow is linear! Since we found no exact expression for $\alpha(K)$, we may estimate it by means of simulation. As a result of the fact that a loss cycle is (for large K) a rare event, it would take much time to obtain an estimate with a high level of confidence. The knowledge of the most likely trajectory to an extreme level enables us to speed up the simulation by *Importance Sampling*. We return to this subject in section 6.

It is rather easy to prove that the ratio of $\pi_i(K)$ and $\alpha(K)$ has a positive upper and lower bound (say M and m , respectively), uniformly in K , see theorem 5.1. This implies

$$\begin{aligned} -\theta &= \lim_{K \rightarrow \infty} \frac{1}{K} \log \alpha(K) = \lim_{K \rightarrow \infty} \frac{1}{K} \log (\alpha(K)m) \\ &\leq \lim_{K \rightarrow \infty} \frac{1}{K} \log \pi_i(K) \\ &\leq \lim_{K \rightarrow \infty} \frac{1}{K} \log (\alpha(K)M) \\ &= \lim_{K \rightarrow \infty} \frac{1}{K} \log \alpha(K) = -\theta. \end{aligned}$$

As a direct consequence, we find that that all inequalities may be replaced by equalities. It is easy to see that the obtained result is weaker than our objective

$$\lim_{K \rightarrow \infty} \pi_i(K) e^{\theta K} = \zeta_i,$$

$i = 1, 2$. Although the theory of Large Deviations provides us useful information about the question how queues build up to a high level, it seems that we cannot obtain results as sharp as desired.

3 Random walk with two absorbing barriers

TO IMPROVE OUR asymptotics of $\pi_i(K)$ ($i = 1, 2$), we first focus on some properties of random walks. Let $\{\xi_n, n \in \mathbb{N}\}$ be a sequence of independent, identically distributed random variables. Fix an initial value $a > 0$. Let the sequence $\{S_n, n \in \mathbb{N}_0\}$ be the partial sums:

$$\begin{aligned} S_0 &:= a, \\ S_n &:= a + \sum_{i=1}^n \xi_i \text{ where } n \in \mathbb{N}. \end{aligned}$$

In other words, we focus on a random walk model starting in a . Our main goal in this section is to obtain an expression for the probability that the random walk $\{S_n, n \in \mathbb{N}_0\}$ hits $[K, \infty)$ before attaining a non-positive value, where K is larger than the initial value a . In the sequel, we call this probability $\alpha_a(K)$. Note that in fact we are considering a random walk model with two absorbing barriers: 0 and K . We assume that $\{S_n, n \in \mathbb{N}_0\}$ has a drift to $-\infty$, so $E(\xi_1) < 0$. To avoid trivialities, it must be possible to reach $[K, \infty)$ with positive probability, so $P(\xi_1 > 0) > 0$. To obtain the desired asymptotics and bounds, we use methods from Martingale theory and Random Walk theory. Basic references in these fields are Williams [1991] and Gut [1987].

In the next definition, we introduce a function, which characterizes the distribution of the increments and which we will use in the sequel intensively.

▷ DEFINITION 3.1

Let $P_{\xi_1}(\cdot)$ denote the cumulative distribution function of ξ_1 . The moment generating function of the random variable ξ_1 is defined as follows:

$$M_{\xi_1}(\theta) := E(\exp(\theta \xi_1)) = \int_{-\infty}^{\infty} e^{\theta x} dP_{\xi_1}(x),$$

for all real θ .

When the moment generating function exists, it contains in some sense all information about the underlying distribution of the random variable. So, it is for instance possible to

deduce the moments (if they exist!) of ξ_1 from $M_{\xi_1}(\cdot)$ by differentiating or evaluating in a Taylor expansion. As we will see, the moment generating function plays a crucial part in the further analysis. Let D_M denote the *domain* of the moment generating function $M_{\xi_1}(\cdot)$:

$$D_M := \{\theta : M_{\xi_1}(\theta) < \infty\}.$$

In the remainder, the equation $M_{\xi_1}(\theta) = 1$ is very important. It appears that under very weak conditions, the existence of a unique positive root can be shown. We present a sufficient condition in the next lemma. Its proof follows standard arguments from convex analysis.

▷ **LEMMA 3.2**

Assume that $M_{\xi_1}(\cdot)$ satisfies

$$\lim_{\theta \rightarrow \partial D_M} M_{\xi_1}(\theta) = \infty,$$

where θ approaches the upper boundary of the domain D_M from the interior D_M° . Then the equation $M_{\xi_1}(\theta) = 1$ has exactly one positive solution for θ .

PROOF. Noting that $\exp(\cdot)$ is a convex function, we have by Jensen's inequality

$$M_{\xi_1}(\theta) = E(\exp(\theta\xi_1)) \geq \exp(\theta E(\xi_1)).$$

From this, we see immediate that if there exists a θ satisfying $M_{\xi_1}(\theta) = 1$, it follows from $E(\xi_1) < 0$ that θ is non-negative.

Since

$$M_{\xi_1}''(\theta) = \int_{-\infty}^{\infty} x^2 e^{\theta x} dP_{\xi_1}(x)$$

is non-negative for all real θ , $M_{\xi_1}(\cdot)$ is convex. According to Rockafellar [1970, page 82], any convex function is continuous on the interior of its domain.

Note that $M_{\xi_1}(0) = 1$ and $M_{\xi_1}'(0) = E(\xi_1) < 0$, so $M_{\xi_1}(\cdot)$ has a negative derivative in $(0, 1)$. From the continuity, we have that $M_{\xi_1}(\theta) < 1$ for some $\theta > 0$.

From the condition, it is clear that arbitrarily large values can be attained for positive θ . Since $M_{\xi_1}(\cdot)$ attains values smaller than 1 as well as values larger than 1, it follows from the continuity of $M_{\xi_1}(\cdot)$ that $M_{\xi_1}(\theta) = 1$ has a positive solution.

We proved the existence of a positive root; now we examine the unicity. Let θ^* be the smallest positive solution of $M_{\xi_1}(\theta) = 1$. Clearly, $M_{\xi_1}'(\theta^*) > 0$. From the convexity, we have that for all $\theta > \theta^*$ the function $M_{\xi_1}(\cdot)$ has a positive slope. We may conclude that $M_{\xi_1}(\theta) > 1$ for all $\theta > \theta^*$. This implies that there exists exactly one positive solution of $M_{\xi_1}(\theta) = 1$. ■

Note that the condition in lemma 3.2 is satisfied in the case that D_M has no upper bound. This can be seen as follows. Since $P(\xi_1 > 0) > 0$, there exists a positive ϵ such

that $P(\xi_1 > \epsilon) > 0$. If D_M has no upper bound, we have the following:

$$\begin{aligned}\lim_{\theta \rightarrow \infty} M_{\xi_1}(\theta) &= \lim_{\theta \rightarrow \infty} \int_{-\infty}^{\infty} e^{\theta x} dP_{\xi_1}(x) \\ &\geq \lim_{\theta \rightarrow \infty} \int_{\epsilon}^{\infty} e^{\theta x} dP_{\xi_1}(x) \\ &\geq \lim_{\theta \rightarrow \infty} e^{\theta \epsilon} \int_{\epsilon}^{\infty} dP_{\xi_1}(x) \\ &= \lim_{\theta \rightarrow \infty} e^{\theta \epsilon} P(\xi_1 > \epsilon) = \infty.\end{aligned}$$

In the previous lemma, we assumed that $M_{\xi_1}(\theta)$ approaches ∞ for θ approaching the upper bound of the domain. Examples of random variables with densities with this property are random variables having an Exponential, Gamma, Erlang or Normal density. Excluded are distributions with extremely long tails, like the Cauchy or LogNormal distribution. In the sequel, we assume throughout that there is one positive number satisfying $M_{\xi_1}(\theta) = 1$, which we will call simply θ .

Define the following stopping time:

$$T_K := \inf\{n \in \mathbb{N}_0 : S_n \geq K \text{ or } S_n \leq 0\}.$$

So T_K is the smallest n at which a value in $[K, \infty)$ or $(-\infty, 0]$ is attained. Note that T_K is indeed a stopping time since the event $\{T_K = n\}$ is independent of $S_{n+1}, S_{n+2}, \dots$. In other words, $\{T_K = n\}$ is completely determined by $S_0, \dots, S_n$.

We can state this a bit more formally. Suppose that we are dealing with the probability triple $(\Omega, \mathcal{F}, P)$. We denote by $\mathcal{F}_n$ the σ -algebra $\sigma\{S_1, \dots, S_n\}$ generated by the random variables $S_1, \dots, S_n$ (where $n \in \mathbb{N}$). Also, let $\mathcal{F}_0$ be the trivial σ -algebra $\{\emptyset, \Omega\}$, where Ω denotes the appropriate sample space. Note that the in this way constructed natural filtration $\{\mathcal{F}_n, n \in \mathbb{N}_0\}$ is an increasing family of σ -algebras, which are sub- σ -algebras of $\mathcal{F}$. We see that the event $\{T_K = n\}$ is measurable with respect to $\mathcal{F}_n$, where $n \in \mathbb{N}_0$. Therefore, T_K is stopping time relative to the family $\{\mathcal{F}_n, n \in \mathbb{N}_0\}$ (cf. Williams [1991, pages 97-101] and Ross [1983, page 229]).

We also see that

$$\alpha_a(K) = P(S_{T_K} \geq K).$$

▷ LEMMA 3.3

T_K is non-defective, i.e., $P(T_K < \infty) = 1$.

In other words: T_K has no probability mass at ∞ .

PROOF. It is possible to find a random variable that T_K dominates, which does not depend on K . Define by T the smallest n at which the process $\{S_n, n \in \mathbb{N}\}$ visits $(-\infty, 0]$. Clearly, $T_K \leq T$ for all K (almost surely). The weak law of large numbers states

that for all positive ϵ ,

$$\lim_{n \rightarrow \infty} P \left(\left| \frac{S_n - a - n E(\xi_1)}{n} \right| > \epsilon \right) = 0.$$

Choosing $\epsilon = -E(\xi_1)/2$ (which is positive!), we have equivalently

$$\lim_{n \rightarrow \infty} P \left(a + \frac{3}{2}n E(\xi_1) \leq S_n \leq a + \frac{1}{2}n E(\xi_1) \right) = 1.$$

If n is larger than $-2a/E(\xi_1)$,

$$P(S_n \leq 0) \geq P \left(a + \frac{3}{2}n E(\xi_1) \leq S_n \leq a + \frac{1}{2}n E(\xi_1) \right),$$

which implies that $\lim_{n \rightarrow \infty} P(S_n \leq 0) = 1$. Note that $\{T = \infty\}$ can be rewritten as the event that $S_i > 0$ for all $i \in \mathbb{N}$. Now it is easy to prove the stated:

$$\begin{aligned} P(T = \infty) &= P \left(\bigcap_{i=0}^{\infty} \{S_i > 0\} \right) = P \left(\lim_{n \rightarrow \infty} \bigcap_{i=0}^n \{S_i > 0\} \right) \\ &= \lim_{n \rightarrow \infty} P \left(\bigcap_{i=0}^n \{S_i > 0\} \right) \leq \lim_{n \rightarrow \infty} P(S_n > 0) = 0, \end{aligned}$$

since $P(\cdot)$ is a continuous set function (Ross [1983, pages 2-3]). We see that T is finite almost surely, and this implies that it also holds that $P(T_K < \infty) = 1$. Note that, as a direct implication of this proof, in the model with only *one* absorbing barrier (the interval $(-\infty, 0]$), this barrier is crossed in the long run with probability 1. ■

The non-defectiveness of T_K implies immediately that S_{T_K} is well-defined and

$$P(S_{T_K} \leq 0) + P(S_{T_K} \geq K) = 1.$$

Note also that $P(S_{T_K} \leq K)$ equals $\alpha_a(K)$, the probability that we are interested in. Using Doob's optional sampling theorem and the observations above, we find the next explicit expression for $\alpha_a(K)$.

▷ THEOREM 3.4

Wald's fundamental equality holds:

$$E \left(\exp(\theta S_{T_K}) \right) = e^{\theta a},$$

where θ denotes the uniquely determined positive solution of $M_{\xi_1}(\theta) = 1$. This implies that

$$\alpha_a(K) = \frac{e^{\theta a} - E \left(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0 \right)}{E \left(\exp(\theta S_{T_K}) \mid S_{T_K} \geq K \right) - E \left(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0 \right)}.$$

PROOF. It is easy to prove that the sequence $\{\exp(\theta S_n), n \in \mathbb{N}\}$ is a *martingale* with respect to its natural filtration (see for instance Williams [1991, pages 93-94]), with value $e^{\theta a}$ in 0. For obvious reasons, we call this kind of martingales *exponential martingales*.

To prove this, we first consider the natural filtration. We define $\mathcal{G}_n$ to be σ -algebra $\sigma\{\exp(\theta S_1), \dots, \exp(\theta S_n)\}$ generated by the random variables $\exp(\theta S_1), \dots, \exp(\theta S_n)$, for $n \in \mathbb{N}$. $\mathcal{G}_0$ is $\{\emptyset, \Omega\}$.

Using the facts that $\exp(\theta S_{n-1})$ is $\mathcal{G}_{n-1}$ -measurable and that ξ_n is independent of $\mathcal{G}_{n-1}$, we see

$$\begin{aligned} E(\exp(\theta S_n) \mid \mathcal{G}_{n-1}) &= E(\exp(\theta S_{n-1}) \exp(\theta \xi_n) \mid \mathcal{G}_{n-1}) \\ &= \exp(\theta S_{n-1}) E(\exp(\theta \xi_n) \mid \mathcal{G}_{n-1}) \\ &= \exp(\theta S_{n-1}) E(\exp(\theta \xi_n)) \\ &= \exp(\theta S_{n-1}), \end{aligned}$$

with probability 1 (for $n \in \mathbb{N}$). This martingale is also studied in Ross [1983, pages 234-236].

It can be seen that, since $\theta \neq 0$, the family $\{\mathcal{F}_n, n \in \mathbb{N}_0\}$ equals $\{\mathcal{G}_n, n \in \mathbb{N}_0\}$. Therefore it holds that T_K is also stopping time relative to $\{\mathcal{G}_n, n \in \mathbb{N}_0\}$.

Trivially, a constant is stopping time with respect to all filtrations. It can be proved that also the minimum of two stopping times is again stopping time (Williams [1991, pages 219-220]). Therefore, we may consider the martingale $\{\exp(\theta S_n), n \in \mathbb{N}_0\}$ with the bounded stopping time $\min(n, T_K)$. According to Williams [1991, page 100] we have by Doob's optimal sampling theorem for martingales equipped with bounded stopping times

$$E(\exp(\theta S_{\min(n, T_K)})) = \exp(\theta S_0) = e^{\theta a},$$

for $n \in \mathbb{N}_0$. Now we want to use Lebesgue's dominated convergence theorem to get the desired result.

- On $\{T_K > n\}$, obviously $0 < E(\exp(\theta S_{\min(n, T_K)})) < e^{\theta K}$.
- On $\{T_K \leq n\}$, we have according to Ross [1974],

$$\begin{aligned} 0 < E(\exp(\theta S_{\min(n, T_K)})) &= E(\exp(\theta S_{T_K})) \\ &\leq E(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0) + \\ &\quad E(\exp(\theta S_{T_K}) \mid S_{T_K} \geq K) \\ &\leq 1 + e^{\theta K} \sup_{r \geq 0} E(\exp(\theta(\xi_1 - r)) \mid \xi_1 \geq r) \\ &< \infty. \end{aligned}$$

In fact, the supremum above should be taken over all non-negative r for which $P(\xi_1 \geq r) > 0$. For notational convenience, we only write $r \geq 0$.

The use of dominated convergence is justified. Interchanging the order of limit and expectation in conjunction with the non-defectiveness of T_K yields:

$$e^{\theta a} = \lim_{n \rightarrow \infty} E(\exp(\theta S_{\min(n, T_K)})) = E(\exp(\theta S_{T_K}))$$

$$= \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0) \times (1 - \alpha_a(K)) + \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \geq K) \times \alpha_a(K),$$

which is just equivalent to the stated. $\blacksquare$

Now we can state the main result of this section. This theorem shows that $\alpha_a(K)$ decays approximately exponentially fast in K . We also deduce a non-trivial upper and lower bound for this probability.

▷ **THEOREM 3.5**

Let θ denote the unique positive solution of $M_{\xi_1}(\theta) = 1$. Then, if ξ_1 has a continuous distribution,

$$\lim_{K \rightarrow \infty} \alpha_a(K) e^{\theta K} = \frac{e^{\theta a} - \lim_{K \rightarrow \infty} \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0)}{\lim_{K \rightarrow \infty} \mathbb{E}(\exp(\theta(S_{T_K} - K)) \mid S_{T_K} \geq K)},$$

a positive constant, say $C(a)$. Furthermore, we have

$$\begin{aligned} \frac{e^{\theta a} - 1}{e^{\theta K} \sup_{r \geq 0} \mathbb{E}(\exp(\theta(\xi_1 - r)) \mid \xi_1 \geq r)} &\leq \alpha_a(K) \\ &\leq \frac{e^{\theta a}}{e^{\theta K} \inf_{r \geq 0} \mathbb{E}(\exp(\theta(\xi_1 - r)) \mid \xi_1 \geq r) - 1}. \end{aligned}$$

PROOF. The upper and lower bound immediately follow from theorem 3.6 and Ross [1974]. Furthermore, clearly

$$\lim_{K \rightarrow \infty} \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0)$$

exists, since then the model with *one* absorbing barrier (which is reached with probability 1, see the proof of lemma 3.3) is involved. Note that this quantity does depend on a . Using lemma 3.7 we proved the stated. $\blacksquare$

Now we treat a useful lemma, which we apply several times in the further analysis, for instance in the proof of lemma 3.7.

▷ **LEMMA 3.6**

Defining θ as the unique positive root of $M_{\xi_1}(\theta) = 1$, and S'_n as the n th partial sum of the random walk starting in 0,

$$\beta(K) := \mathbb{P}(\exists n \in \mathbb{N} : S'_n \geq K) \leq e^{-\theta K}.$$

PROOF. We give two easy derivations of this lemma. The first is due to Ross [1974, 1983].

- Define stopping time $T_{K,L}$ for positive K and negative L :

$$T_{K,L} := \inf\{n \in \mathbb{N} : S'_n \geq K \text{ or } S'_n \leq L\}.$$

We see immediately that $\{S'_n, n \in \mathbb{N}_0\}$ is martingale, 1 in 0. The optional sampling theorem says:

$$1 = \mathbb{E}\left(\exp(\theta S'_{T_{K,L}}) \mid S'_{T_{K,L}} \leq L\right) \times \mathbb{P}(S'_{T_{K,L}} \leq L) + \\ \mathbb{E}\left(\exp(\theta S'_{T_{K,L}}) \mid S'_{T_{K,L}} \geq K\right) \times \mathbb{P}(S'_{T_{K,L}} \geq K).$$

Obviously, the first conditional expectation is smaller than $e^{\theta L}$, so its limit for $L \rightarrow -\infty$ equals 0. The second conditional expectation is larger than $e^{\theta K}$. Trivially, $\beta(K)$ equals $\mathbb{P}(S'_{T_{K,L}} \geq K)$ for L approaching $-\infty$. We conclude that $\beta(K)$ is dominated by $e^{-\theta K}$ and we are done.

- Doob's inequality for non-negative (sub-)martingales (see Williams [1991, page 137]) states that

$$c \times \mathbb{P}\left(\sup_{i \in \{0, \dots, n\}} \exp(\theta S'_i) \geq c\right) \leq \mathbb{E}\left(\exp(\theta S'_n)\right) = 1,$$

for all positive c and $n \in \mathbb{N}$. It follows immediately that the proof is completed:

$$\beta(K) = \mathbb{P}\left(\sup_{i \in \mathbb{N}} \exp(\theta S'_i) \geq e^{\theta K}\right) \leq e^{-\theta K}.$$

Note that it also an exponential lower bound can be found:

$$\beta(K) \geq \frac{e^{-\theta K}}{\sup_{r \geq 0} \mathbb{E}\left(\exp(\theta(\xi_1 - r)) \mid \xi_1 \geq r\right)},$$

see Ross [1974]. We also see that (in for itself speaking notation)

$$\beta(K) \text{ and } \mathbb{E}\left(\exp(\theta S'_{T_{K,-\infty}}) \mid S'_{T_{K,-\infty}} \geq K\right)$$

are reciprocals. In Feller [1971, pages 374-377 and 406-407] it can be found that $\beta(K) e^{\theta K}$ converges to a constant. Hence we found also that

$$\lim_{K \rightarrow \infty} \mathbb{E}\left(\exp(\theta(S'_{T_{K,-\infty}} - K)) \mid \exists n : S'_n \geq K\right)$$

is a positive constant. We will use this result in the proof of the next lemma. ■

We may conclude from this lemma, that if the random walk started in 0, the probability of ever exceeding K is dominated by $e^{-\theta K}$. We see that this probability approaches 0 when $K \rightarrow \infty$. This implies that for all $\epsilon > 0$ we can find a level K , such that the probability

that the random walk ever crosses K is smaller than ϵ . This property of random walks with drift to $-\infty$ will be used several times later on.

▷ **LEMMA 3.7**

The following limit exists:

$$\lim_{K \rightarrow \infty} \mathbb{E} \left(\exp(\theta(S_{T_K} - K)) \mid S_{T_K} \geq K \right).$$

PROOF. This result can be derived in four steps.

- Weaken the condition $\{S_{T_K} \geq K\}$ to $\{\exists n \in \mathbb{N}_0 : S_n \geq K\}$. Let γ_K be the quantity by which K is exceeded by $\{S_n, n \in \mathbb{N}_0\}$. In the proof of the previous lemma, we already saw that

$$\mathbb{E} \left((\exp(\theta \gamma_K) \mid \exists n : S_n \geq K) \right)$$

is a positive constant.

In the next three steps, we prove that the weakening of the condition

$$\{S_{T_K} \geq K\} \quad \text{to} \quad \{\exists n : S_n \geq K\}$$

does not influence the existence and the value of the limit.

- In this step we verify that under the condition that K is exceeded before attaining a value in $(-\infty, 0]$, there exists (letting $K \rightarrow \infty$) an $i \leq T_K$ such that $S_i \in (K/3, 2K/3)$ with probability 1.

Since $\alpha_a(K) \geq \beta(K)/C$, uniformly in K , for some positive constant C (theorem 3.5 and lemma 3.6),

$$\begin{aligned} & \limsup_{K \rightarrow \infty} \mathbb{P} \left(\exists i \leq T_K : K/3 < S_i < 2K/3 \mid S_{T_K} \geq K \right) \\ & \leq \limsup_{K \rightarrow \infty} \frac{\mathbb{P}(\gamma_{K/3} \geq K/3, \exists n \in \mathbb{N} : S_n \geq K)}{\mathbb{P}(S_{T_K} \geq K)} \\ & \leq C \limsup_{K \rightarrow \infty} \mathbb{P}(\gamma_{K/3} \geq K/3 \mid \exists n \in \mathbb{N} : S_n \geq K). \end{aligned}$$

Just as in Ross [1974], conditioning yields

$$\mathbb{E}(\gamma_{K/3} \geq K/3 \mid \exists n : S_n \geq K) \leq \sup_{r \geq 0} \mathbb{E} \left(\xi_1 - r \mid \xi_1 \geq r, \exists n : \sum_{i=1}^n \xi_i \geq 2K/3 + r \right).$$

From the bounds obtained in lemma 3.6, we conclude that there exists a positive constant C' such that for $x \geq r$,

$$d\mathbb{P}_{\xi_1} \left(x \mid \xi_1 \geq r, \exists n : \sum_{i=1}^n \xi_i \geq 2K/3 + r \right)$$

$$\begin{aligned}
&= dP_{\xi_1}(x) \beta \left(\frac{2K}{3} + r - x \right) / \left(\int_r^\infty \beta \left(\frac{2K}{3} + r - y \right) dP_{\xi_1}(y) \right) \\
&\leq C' dP_{\xi_1}(x) e^{\theta x} / \left(\int_r^\infty e^{\theta y} dP_{\xi_1}(y) \right) = C' dP_{\xi'_1}(x | \xi'_1 \geq r).
\end{aligned}$$

Here the random variable in the ξ'_1 in the previous display has distribution function

$$P(\xi'_1 \leq u) = \int_{-\infty}^u e^{\theta x} dP_{\xi_1}(x),$$

i.e., the tilted or twisted version of the distribution of ξ_1 . This important distribution will play an important part several times later on. Now we use Markov's inequality for non-negative random variables:

$$\begin{aligned}
&\limsup_{K \rightarrow \infty} P(\gamma_{K/3} \geq K/3 \mid \exists n : S_n \geq K) \\
&\leq \limsup_{K \rightarrow \infty} \frac{E(\gamma_{K/3} \geq K/3 \mid \exists n : S_n \geq K)}{K/3} \\
&\leq \limsup_{K \rightarrow \infty} \frac{C' \sup_{r \geq 0} E(\xi'_1 - r \mid \xi'_1 \geq r)}{K/3} = 0,
\end{aligned}$$

which terminates the proof of this step.

Hence, we found that (with probability 1) a record value is attained in $(K/3, 2K/3)$ (when $K \rightarrow \infty$).

- We call the smallest record value larger than $K/3$ (random variable) R . Define

$$\begin{aligned}
T_{a,b} &:= \inf \{n \in \mathbb{N} : S_n \geq b, \text{ starting in } a\}, \\
T'_{a,b} &:= \inf \{n \in \mathbb{N} : S_n \leq b, \text{ starting in } a\}.
\end{aligned}$$

Then we have according to lemma 3.6 that holds with probability 1,

$$\begin{aligned}
&P(T'_{R,0} < T_{R,K} \mid \text{starting in } R, \exists n \in \mathbb{N} : S_n \geq K) \\
&\leq \frac{P(T'_{R,0} < T_{R,K}, \text{ starting in } R, \exists n \in \mathbb{N} : S_n \geq K)}{\beta(K - R)} \\
&\leq \frac{\beta(K)}{\beta(K - R)} \leq \frac{e^{-\theta K}}{C e^{-\theta(K-R)}} = \frac{1}{C} e^{-\theta R},
\end{aligned}$$

for some positive constant C . If K approaches ∞ , $R \in (K/3, 2K/3)$ almost surely. It follows that

$$\lim_{K \rightarrow \infty} P(T'_{R,0} < T_{R,K} \mid \text{starting in } R, \exists n \in \mathbb{N} : S_n \geq K) = 0.$$

We found that the probability that K is reached visiting $(-\infty, 0]$ (given that K is reached) is negligible if K grows large, provided that we start in a value large enough.

- So, the probability that (starting in R) the interval $[K, \infty)$ is reached without attaining values in $(-\infty, 0]$, given that $[K, \infty)$ is reached, goes to 1 when $K \rightarrow \infty$. It follows that, starting in R instead of a , we may weaken the condition $\{S_{T_K} \geq K\}$ to $\{\exists n : S_n \geq K\}$. This can be explained as follows. Define the following events:

$$\begin{aligned} C_{K,1} &:= \{S_{T_K} \geq K\} \\ C_{K,2} &:= \{\exists n : S_n \geq K\}. \end{aligned}$$

It is clear that $C_{K,1} \subseteq C_{K,2}$ (for all K). We proved in the third step that

$$\lim_{K \rightarrow \infty} \mathbb{P}(C_{K,1} | C_{K,2}) = 1.$$

We leave it to the reader to verify that these two relations yield

$$\begin{aligned} \text{(i)} \quad & \lim_{K \rightarrow \infty} \frac{\mathbb{P}(C_{K,1})}{\mathbb{P}(C_{K,2})} = 1; \\ \text{(ii)} \quad & \lim_{K \rightarrow \infty} \frac{\mathbb{P}(C_{K,2} \setminus C_{K,1})}{\mathbb{P}(C_{K,2})} = 0. \end{aligned}$$

We define $A_K := \{\gamma_K > u\}$. We already saw that $\lim_{K \rightarrow \infty} \mathbb{P}(A_K | C_{K,2})$ exists, starting in a (see step 1). Note that this limit does not depend on a . Recalling that $R < 2K/3$, it follows that this limit also exists starting in (the random value) R instead of a . But starting in R we have

$$\begin{aligned} \lim_{K \rightarrow \infty} \mathbb{P}(A_K | C_{K,2}) &= \lim_{K \rightarrow \infty} \frac{\mathbb{P}(A_K \cap C_{K,2})}{\mathbb{P}(C_{K,2})} \\ &\stackrel{\text{(ii)}}{=} \lim_{K \rightarrow \infty} \left(\frac{\mathbb{P}(A_K \cap C_{K,2})}{\mathbb{P}(C_{K,2})} - \frac{\mathbb{P}(A_K \cap (C_{K,2} \setminus C_{K,1}))}{\mathbb{P}(C_{K,2})} \right) \\ &= \lim_{K \rightarrow \infty} \frac{\mathbb{P}(A_K \cap C_{K,1})}{\mathbb{P}(C_{K,2})} \\ &\stackrel{\text{(i)}}{=} \lim_{K \rightarrow \infty} \frac{\mathbb{P}(A_K \cap C_{K,1})}{\mathbb{P}(C_{K,1})} \\ &= \lim_{K \rightarrow \infty} \mathbb{P}(A_K | C_{K,1}) \end{aligned}$$

We find that the limit of $\mathbb{P}(A_K | C_{K,1})$ exists, just as desired. ■

Bearing in mind that $\beta(K)e^{\theta K}$ converges to a certain positive constant, say C , we find the asymptotic distribution for the increments of the random walk on the path to level K :

$$\begin{aligned} & \lim_{K \rightarrow \infty} \mathbb{P}(\xi_1 > u | \exists n \in \mathbb{N} : S_n \geq K) = \\ & \lim_{K \rightarrow \infty} \int_{(u, K)} \frac{\beta(K-x)}{\beta(K)} d\mathbb{P}_{\xi_1}(x) + \lim_{K \rightarrow \infty} \int_{[K, \infty)} \frac{1}{\beta(K)} d\mathbb{P}_{\xi_1}(x). \end{aligned}$$

Now we note that

$$\frac{\beta(K-x)}{\beta(K)} \leq \frac{e^{\theta x}}{\sup_{r \geq 0} \mathbb{E}(\exp(\theta(\xi - r)) \mid \xi_1 \geq r)} \text{ and } \int_{(u,K)} e^{\theta x} dP_{\xi_1}(x) \leq 1,$$

dominated convergence applies to the first limit. It follows that the interchanging of limit and integral is allowed, and we find

$$\begin{aligned} \int_{(u,\infty)} \lim_{K \rightarrow \infty} 1_{(u,K)}(x) \frac{\beta(K-x)}{\beta(K)} dP_{\xi_1}(x) &= \\ \int_{(u,\infty)} \lim_{K \rightarrow \infty} 1_{(u,K)}(x) \frac{\beta(K-x)e^{\theta(K-x)}}{\beta(K)e^{\theta K}} e^{\theta x} dP_{\xi_1}(x) &= \int_{(u,\infty)} e^{\theta x} dP_{\xi_1}(x). \end{aligned}$$

It is very easy to prove that the second limit equals 0 (this problem can be tackled in the same way as in the proof of theorem 4.4). We see that under the condition $\{\exists n : S_n \geq K\}$ the ξ_i seem to be sampled from a distribution with cumulative distribution function

$$\int_{-\infty}^u e^{\theta x} dP_{\xi_1}(x) \text{ instead of } \int_{-\infty}^u dP_{\xi_1}(x),$$

for $K \rightarrow \infty$. Approximately, the same holds under $\{S_{T_K} \geq K\}$ and K tending to ∞ . This observation gives rise to the change of measure used in the fast simulations described in section 6.

4 The probability of a loss cycle

TO MEET OUR objectives in queueing, we have to modify our model slightly. For initial value $a > 0$ we define

$$\begin{aligned} S_0 &:= a \\ S_n &:= a + \sum_{i=1}^n \xi_i \text{ where } n \in \mathbb{N}, \end{aligned}$$

with $\xi_i \stackrel{d}{=} -A$ if i is odd and $\xi_i \stackrel{d}{=} B$ if i is even, for some non-negative random variables A and B having a continuous distribution. Let the family $\{\xi_i, i = 1, 2, \dots\}$ consist of independent random variables. We see that the summands are not identically distributed anymore. Later on in this section, we will choose A and B such that this model covers the queueing model considered in section 1.

We assume throughout that $P(B > A) > 0$, and $E(A) > E(B)$. Let θ be the unique positive solution of $E(e^{-\theta A})E(e^{\theta B}) = 1$. Define

$$U_n := (\exp(\theta S_n)) \times (1_{\{n \text{ even}\}} + E(e^{\theta B})1_{\{n \text{ odd}\}}).$$

In lemma 4.1 and theorem 4.2 we present results parallel to results obtained in section 3 for the standard random walk.

▷ LEMMA 4.1

$\{U_n, n \in \mathbb{N}_0\}$ is martingale with respect to its own filtration, with value $e^{\theta a}$ in 0.

PROOF. The reasoning behind this lemma is straightforward. Construct the natural filtration as before (see theorem 3.4). First, we assume $n \in \mathbb{N}$ to be odd. Almost surely,

$$\begin{aligned} E(U_n \mid \sigma\{U_0, \dots, U_{n-1}\}) &= E(\exp(\theta S_n) E(e^{\theta B}) \mid S_{n-1}) \\ &= \exp(\theta S_{n-1}) E(\exp(\theta \xi_n)) E(e^{\theta B}) \\ &= \exp(\theta S_{n-1}) E(e^{-\theta A}) E(e^{\theta B}) \\ &= \exp(\theta S_{n-1}) = U_{n-1}. \end{aligned}$$

In similar fashion, if $n \in \mathbb{N}$ is even, with probability 1,

$$\begin{aligned} E(U_n \mid \sigma\{U_0, \dots, U_{n-1}\}) &= E(\exp(\theta S_n) \mid S_{n-1}) \\ &= \exp(\theta S_{n-1}) E(\exp(\theta \xi_n)) \\ &= \exp(\theta S_{n-1}) E(e^{\theta B}) = U_{n-1}. \end{aligned}$$

It is clear that $\{U_n, n \in \mathbb{N}_0\}$ is a martingale. Note that the U_n can also be regarded as the product of n independent unit mean random variables. It is easy to prove that all sequences with this property are martingales (Williams [1991, page 95]). ■

In the same way as in the previous section we define stopping time T_K as the first epoch at which the process $\{S_n, n = 0, 1, \dots\}$ leaves the open interval $(0, K)$. Analogously, we can prove that Wald's fundamental equality holds:

$$E(U_{T_K}) = U_0 = e^{\theta a}.$$

Again, let $\alpha_a(K)$ be the probability that, starting in a , a value in $[K, \infty)$ is attained by $\{S_n, n = 0, 1, \dots\}$, before $(-\infty, 0]$ is hit. It now follows that

$$\begin{aligned} e^{\theta a} &= E(U_{T_K}) \\ &= E(U_{T_K} \mid S_{T_K} \geq K) \times \alpha_a(K) + \\ &\quad E(U_{T_K} \mid S_{T_K} \leq 0) \times (1 - \alpha_a(K)). \end{aligned}$$

Note that under the condition $\{S_{T_K} \geq K\}$, T_K is even with probability 1, whereas under $\{S_{T_K} \leq 0\}$, $P(T_K \text{ is odd}) = 1$. Using the property 'taking out what is known' of the conditional expectation (see Williams [1991, page 88]), we get the following equality:

$$\begin{aligned} e^{\theta a} &= E(\exp(\theta S_{T_K}) \mid S_{T_K} \geq K) \times \alpha_a(K) + \\ &\quad E(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0) E(e^{\theta B}) \times (1 - \alpha_a(K)). \end{aligned}$$

Hence we proved the equivalent of theorem 3.4,

▷ THEOREM 4.2

$\alpha_a(K)$ equals the ratio of $N_a(K)$ and $D_a(K)$, where the numerator $N_a(K)$ is defined by

$$e^{\theta a} - \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0) \mathbb{E}(e^{\theta B})$$

and the denominator $D_a(K)$ by

$$\mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \geq K) - \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0) \mathbb{E}(e^{\theta B}).$$

Analogously to the results derived in the previous section, one can prove that $\alpha_a(K)$ tends approximately exponentially fast to 0 when K grows large:

$$\lim_{K \rightarrow \infty} \alpha_a(K) e^{\theta K} = \frac{e^{\theta a} - \lim_{K \rightarrow \infty} \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0) \mathbb{E}(e^{\theta B})}{\lim_{K \rightarrow \infty} \mathbb{E}(\exp(\theta(S_{T_K} - K)) \mid S_{T_K} \geq K)},$$

a constant, say $C(a)$. Just as in the case of the independent, identically distributed increments, we can find a non-trivial upper and lower bound for $\alpha_a(K)$. It is left to the reader to verify that for all positive K

$$\begin{aligned} C_A(L) := \inf_{r \geq 0} \mathbb{E}(e^{\theta(r-A)} \mid A \geq r) &\leq \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \leq 0) \\ &\leq \sup_{r \geq 0} \mathbb{E}(e^{\theta(r-A)} \mid A \geq r) =: C_A(U) \end{aligned}$$

and on the other hand

$$\begin{aligned} C_B(L) := \inf_{r \geq 0} \mathbb{E}(e^{\theta(B-r)} \mid B \geq r) &\leq \mathbb{E}(\exp(\theta S_{T_K}) \mid S_{T_K} \geq K) e^{-\theta K} \\ &\leq \sup_{r \geq 0} \mathbb{E}(e^{\theta(B-r)} \mid B \geq r) =: C_B(U), \end{aligned}$$

see also Ross [1974] (note that this result is achieved by conditioning on T_K and S_{T_K-1}). We also see that $C_A(U)$ and $C_A(L)$ are not larger than 1, whereas $C_B(U)$ and $C_B(L)$ are equal to or larger than 1. Noting that it is easy to prove (cf. theorem 3.5) that the limits of $N_a(K)$ and $D_a(K)$ exist ($K \rightarrow \infty$), we gave the proof of the next theorem:

▷ THEOREM 4.3

The probability $\alpha_a(K)$ is asymptotically exponential:

$$\lim_{K \rightarrow \infty} \alpha_a(K) e^{\theta K} = C(a),$$

where $C(a)$ is a positive constant. Uniformly in

$$K > K_0 := \max \left\{ \frac{1}{\theta} \log \left(\mathbb{E}(e^{\theta B}) \frac{C_A(U)}{C_B(L)} \right), 0 \right\}$$

we have the following upper and lower bound for $\alpha_a(K)$:

$$\frac{e^{\theta a} - \mathbb{E}(e^{\theta B}) C_A(U)}{e^{\theta K} C_B(U) - \mathbb{E}(e^{\theta B}) C_A(L)} \leq \alpha_a(K) \leq \frac{e^{\theta a} - \mathbb{E}(e^{\theta B}) C_A(L)}{e^{\theta K} C_B(L) - \mathbb{E}(e^{\theta B}) C_A(U)}.$$

As explained in section 2, in the GI/G/1 queueing model, $A \stackrel{d}{=} \sigma(A_1 - A_0)$ and $B \stackrel{d}{=} B_1$. If $\alpha(K)$ denotes the probability that in a cycle a customer is rejected, it is easy to verify that we have

$$\alpha(K) = \int_0^K \alpha_a(K) dB(a) + \int_K^\infty dB(a),$$

invoking the law of total probability, conditioning on the work requirement of the first customer. To guarantee that we may use the results obtained earlier, we assume that θ is the uniquely determined positive root of the equation

$$E(e^{-\sigma\theta(A_1 - A_0)})E(e^{\theta B_1}) = \int_0^\infty e^{-\sigma\theta x} dA(x) \int_0^\infty e^{\theta x} dB(x) = 1.$$

▷ THEOREM 4.4

The probability of a loss cycle is asymptotically exponential:

$$\lim_{K \rightarrow \infty} \alpha(K)e^{\theta K} = \int_0^\infty C(a) dB(a) < \infty,$$

a finite constant. Uniformly in $K > K_0$,

$$\begin{aligned} \int_0^K \frac{e^{\theta a} - E(e^{\theta B})C_A(U)}{e^{\theta K}C_B(U) - E(e^{\theta B})C_A(L)} dB(a) + \int_K^\infty dB(a) &\leq \alpha(K) \\ &\leq \int_0^K \frac{e^{\theta a} - E(e^{\theta B})C_A(L)}{e^{\theta K}C_B(L) - E(e^{\theta B})C_A(U)} dB(a) + \int_K^\infty dB(a). \end{aligned}$$

PROOF. Using theorem 4.3, the second part of the stated is trivial. Consider the first part. Obviously, conditioning on the amount of work brought along by the first entering customer, yields

$$\lim_{K \rightarrow \infty} \alpha(K)e^{\theta K} = \lim_{K \rightarrow \infty} \int_0^K \alpha_a(K)e^{\theta K} dB(a) + \lim_{K \rightarrow \infty} \int_K^\infty e^{\theta K} dB(a),$$

if these limits exist. Now we show that the first limit equals a positive constant, whereas the second equals 0.

- We first investigate

$$\lim_{K \rightarrow \infty} \int_0^K \alpha_a(K)e^{\theta K} dB(a).$$

Using the upper bound derived in theorem 4.3, we see that for K larger than $\log(2E(e^{\theta B}))/\theta$ it holds that

$$\alpha_a(K)e^{\theta K} \leq \frac{e^{\theta a} e^{\theta K}}{e^{\theta K} - E(e^{\theta B})} \leq 2e^{\theta a}.$$

Since the product of the moment generating functions of the random variables $-\sigma(A_1 - A_0)$ and B_1 is finite in θ (namely 1), it is clear that

$$\int_0^\infty 2e^{\theta a} dB(a) = 2E(e^{\theta B_1}) < \infty.$$

We found a dominating integrable function for $\alpha_a(K)e^{\theta K}$, independent of K . Therefore, we have by invoking Lebesgue's dominated convergence theorem and theorem 4.3

$$\begin{aligned}\lim_{K \rightarrow \infty} \int_0^K \alpha_a(K) e^{\theta K} dB(a) &= \lim_{K \rightarrow \infty} \int_0^\infty 1_{[0,K)}(a) \alpha_a(K) e^{\theta K} dB(a) \\ &= \int_0^\infty \lim_{K \rightarrow \infty} 1_{[0,K)}(a) \alpha_a(K) e^{\theta K} dB(a) \\ &= \int_0^\infty C(a) dB(a),\end{aligned}$$

a finite constant.

- Finally we examine the second limit

$$\lim_{K \rightarrow \infty} \int_K^\infty e^{\theta K} dB(a).$$

Obviously, on $\{a \geq K\}$ the integrand $e^{\theta K}$ is dominated by $e^{\theta a}$, so

$$\lim_{K \rightarrow \infty} \int_K^\infty e^{\theta K} dB(a) \leq \lim_{K \rightarrow \infty} \int_K^\infty e^{\theta a} dB(a) = 0,$$

since $E(\exp(\theta B_1)) < \infty$, as mentioned before. This completes the proof.

We may conclude that the probability of a loss cycle approximately obeys an exponential decay in buffersize K . ■

We derived the desired properties (see section 1) for $\alpha(K)$ instead of $\pi_1(K)$ and $\pi_2(K)$. It is clear that these quantities are strongly related. In the next section we study this relation.

5 The relation between $\pi_i(K)$ and $\alpha(K)$

IN THIS SECTION we investigate the connection between $\alpha(K)$ and $\pi_i(K)$, where $i = 1, 2$. We recall that $\alpha(K)$ is the same for both models. Define:

$$\begin{aligned}L_K^{(i)}(t) &:= \text{Number of customers rejected in } [0, t]; \\ A_K^{(i)}(t) &:= \text{Number of customers arrived in } [0, t]; \\ L_K^{(i)} &:= \text{Number of customers rejected in a cycle}; \\ A_K^{(i)} &:= \text{Number of customers arrived in a cycle}; \\ C_K^{(i)} &:= \text{Length (in time) of a cycle}.\end{aligned}$$

For the notion 'cycle', we refer to section 2. The subscript K indicates the buffer capacity, whereas the superscript i corresponds with the queueing discipline: $i = 1$ denotes the

partial rejection case and $i = 2$ the complete rejection case. Using the *Renewal-Reward theorem* (cf. for instance Tijms [1986, page 10] or Wolff [1989, page 60]),

$$\lim_{t \rightarrow \infty} \frac{L_K^{(i)}(t)}{t} = \frac{\mathbb{E}(L_K^{(i)})}{\mathbb{E}(C_K^{(i)})}$$

and similarly

$$\lim_{t \rightarrow \infty} \frac{A_K^{(i)}(t)}{t} = \frac{\mathbb{E}(A_K^{(i)})}{\mathbb{E}(C_K^{(i)})}$$

with probability 1 ($i = 1, 2$). We find, using the fact that almost sure convergence survives continuous transformations,

$$\pi_i(K) := \lim_{t \rightarrow \infty} \frac{L_K^{(i)}(t)}{A_K^{(i)}(t)} = \frac{\mathbb{E}(L_K^{(i)})}{\mathbb{E}(A_K^{(i)})},$$

almost surely ($i = 1, 2$). Recalling that a loss cycle is defined as a cycle in which at least one customer is (partially or completely) rejected, we get by conditioning on the occurrence of a loss cycle

$$\begin{aligned} \pi_i(K) &= \frac{\mathbb{E}(L_K^{(i)} \mid \text{no loss cycle}) \mathbb{P}_K(\text{no loss cycle})}{\mathbb{E}(A_K^{(i)})} + \\ &\quad \frac{\mathbb{E}(L_K^{(i)} \mid \text{loss cycle}) \mathbb{P}_K(\text{loss cycle})}{\mathbb{E}(A_K^{(i)})}, \end{aligned}$$

which reduces to

$$\frac{\pi_i(K)}{\alpha(K)} = \frac{\mathbb{E}(L_K^{(i)} \mid \text{loss cycle})}{\mathbb{E}(A_K^{(i)})},$$

$i = 1, 2$. One of our objectives was (see section 1) to find non-trivial functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ such that

$$\eta_1(K) \leq \pi_i(K) \leq \eta_2(K),$$

for $i = 1, 2$ and K large enough. Using theorem 4.4 and the result obtained above, we only have to prove that the ratio of $\pi_i(K)$ (where $i = 1, 2$) and $\alpha(K)$ has a positive upper and lower bound, uniformly in K . Then we found, in case of $A_1 - A_0$ and B_1 having at least exponentially fast to 0 tending tails, functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ with the desired property.

▷ THEOREM 5.1

There exist two positive constants m and M such that

$$m \leq \frac{\pi_i(K)}{\alpha(K)} = \frac{\mathbb{E}(L_K^{(i)} \mid \text{loss cycle})}{\mathbb{E}(A_K^{(i)})} \leq M,$$

$i = 1, 2$ and $K > 0$.

PROOF. It is sufficient to prove that $E(L_K^{(i)} \mid \text{loss cycle})$ as well as $E(A_K^{(i)})$ has a positive upper and lower bound for $i = 1, 2$.

- First consider $E(L_K^{(i)} \mid \text{loss cycle})$. Let $\{\xi_n, n \in \mathbb{N}\}$ be a family of independent, identically distributed random variables with

$$\xi_1 \stackrel{d}{=} -\sigma(A_1 - A_0) + B_1.$$

The sequence $\{S_n, n \in \mathbb{N}_0\}$ denotes the partial sums of the ξ_n . Put $S_0 := 0$. We leave it to the reader to verify that

$$1 \leq E(L_K^{(i)} \mid \text{loss cycle}) \leq 1 + E(X),$$

with $P(X = j) = p^j(1 - p)$ ($j \in \mathbb{N}_0$). Here p denotes

$$P(\{S_n, n \in \mathbb{N}\} \text{ hits } [0, \infty)),$$

i.e., the probability that $\{S_n, n \in \mathbb{N}\}$ attains at least once a value in $[0, \infty)$. Since $E(\xi_1) < 0$, it is a well-known result from random walk theory (see for instance Feller [1971, pages 396-397]) that $p < 1$. But

$$1 + E(X) = 1 + \sum_{j=1}^{\infty} j p^j (1 - p) < 1 + \sum_{j=1}^{\infty} j p^j = 1 + \frac{p}{(1 - p)^2} < \infty.$$

Note that this upper bound cannot be calculated in practice, since we omit a simple expression for the probability p . Fortunately, we can give a bit more tractable upper bound for the mean of $L_K^{(i)}$, conditioned on a loss cycle. According to Wolff [1989, page 418],

$$\log \left(\frac{1}{1 - p} \right) = \sum_{n=1}^{\infty} \frac{P(S_n > 0)}{n}.$$

Recalling definition 3.1, we define the so-called *Large Deviations rate function* or the *Legendre-Fenchel transform* of the function $\log M_{\xi_1}(\cdot)$:

$$I(x) := \sup_{\theta} (\theta x - \log M_{\xi_1}(\theta)).$$

The following result involving sample averages is due to Chernoff:

$$P(S_n > 0) \leq P(S_n \geq 0) = P(S_n/n \geq 0) \leq e^{-nI(0)},$$

uniformly in $n \in \mathbb{N}$. It can be seen that $E(\xi_1) < 0$ implies that $I(0) > 0$. These results can be found in for instance Chernoff [1952], Ellis [1985, page 247] or Bucklew [1990, pages 117-119].

It follows immediately that

$$\log \left(\frac{1}{1-p} \right) \leq \sum_{n=1}^{\infty} \frac{e^{-nI(0)}}{n} = \log \left(\frac{1}{1-e^{-I(0)}} \right),$$

which leads to $p \leq \exp(-I(0)) < 1$.

An upper bound for $E(L_K^{(i)} \mid \text{loss cycle})$ becomes

$$1 + \frac{e^{-I(0)}}{(1-e^{-I(0)})^2},$$

which can be calculated easily. Note that no explicit expression for the density or cumulative distribution function of ξ_1 is required, since (in for itself speaking notation) we have $M_{\xi_1}(\theta) = M_{A_1-A_0}(-\sigma\theta)M_{B_1}(\theta)$.

- Now we investigate $E(A_K^{(i)})$. It is immediately clear that for all positive K this quantity is not smaller than 1. Also, the number of customers arrived in a cycle will be smaller than in a queueing system without capacity restriction. That is,

$$E(A_K^{(i)}) \leq E(A_\infty) := \lim_{K \rightarrow \infty} E(A_K^{(i)}) < \infty,$$

$i = 1, 2$. An expression for the generating function of the random variable A_∞ can be found in Wolff [1989, pages 420-422]. Using this relation, an expression for $E(A_\infty)$ can be derived.

Note that the bounds obtained above are valid for partial as well as complete rejection. ■

Our second goal was to prove that for some positive constants θ , ζ_1 and ζ_2 it holds that

$$\lim_{K \rightarrow \infty} \pi_i(K) e^{\theta K} = \zeta_i,$$

for $i = 1, 2$. According to theorem 4.4, we only have to prove that the ratio of $\pi_i(K)$ ($i = 1, 2$) and $\alpha(K)$ goes to a positive constant as K approaches ∞ . From observations earlier in this section, it is sufficient to prove that

$$\lim_{K \rightarrow \infty} E(L_K^{(i)} \mid \text{loss cycle}) \quad \text{and} \quad \lim_{K \rightarrow \infty} E(A_K^{(i)})$$

exist ($i = 1, 2$).

In the proof of theorem 5.1, we already saw that the existence of the limit of $E(A_K^{(i)})$ for $K \rightarrow \infty$ is guaranteed. It takes only a little effort to prove that

$$\lim_{K \rightarrow \infty} E(L_K^{(1)} \mid \text{loss cycle})$$

exists, since it is easy to verify that $E(L_K^{(1)} \mid \text{loss cycle})$ is bounded (see theorem 5.1) and increasing in K . On the contrary, considering the queueing discipline with complete rejection, we need a lot of tedious reasoning.

In the remainder of this section, we only consider the complete rejection case. We use the same model as in section 4: $S_0 := 0$ and S_n is defined to be the sum of the first n (where $n \in \mathbb{N}$) ξ_i . Here $\xi_i \stackrel{d}{=} B_1$ if i is odd and $\xi_i \stackrel{d}{=} -\sigma(A_1 - A_0)$ if i is even.

Again, define T_K by the first epoch at which the interval $[0, K]$ is left by the stochastic process. Just as in lemma 3.9 concerning $(S_{T_K} - K \mid S_{T_K} \geq K)$ (the ‘overshoot’), one can prove that $(K - S_{T_K-1} \mid S_{T_K} \geq K)$ (the ‘undershoot’) has a limiting distribution when K goes to infinity. Define

$$P_K(x) := P(K - S_{T_K-1} \leq x \mid S_{T_K} \geq K) \quad \text{where } x \in [0, K],$$

and its limit

$$P(x) := \lim_{K \rightarrow \infty} P_K(x) \quad \text{where } x \in [0, \infty).$$

Note that, for fixed x , the mean number of rejected customers after the first rejection and before the end of the cycle, given that the amount of work in the system is $K - x$ at the first rejection, is bounded (theorem 5.1!) and increases in K . We may conclude that this quantity, which we abbreviate to $E(L_K; x)$, approaches a limit value when $K \rightarrow \infty$. We will call this pointwise limit $E(L_\infty; x)$. In the sequel we investigate

$$E(L_K^{(2)} \mid \text{loss cycle}) = 1 + \int_0^K E(L_K; x) dP_K(x)$$

for $K \rightarrow \infty$. The candidate for the limit is

$$E(L_\infty^{(2)} \mid \text{loss cycle}) := 1 + \int_0^\infty E(L_\infty; x) dP(x).$$

Before we prove that this limit exists, we first treat three lemmas.

▷ LEMMA 5.2

$E(L_K; x)$ is a continuous function of x , where $x \in [0, K]$.

PROOF. Fix an $\epsilon > 0$ and an $x \in [0, K]$. We have to prove that choosing x' near enough to x (but $x' \neq x$) leads to

$$\left| E(L_K; x) - E(L_K; x') \right| < \epsilon.$$

Without loss of generality, we may assume $x' < x$. In for itself speaking notation we find the following upper bound, using the triangle inequality:

$$\begin{aligned} \left| E(L_K; x) - E(L_K; x') \right| &= \left| \sum_{i=1}^{\infty} i P(L_x(K) = i) - \sum_{i=1}^{\infty} i P(L_{x'}(K) = i) \right| \\ &\leq \sum_{i=1}^{\infty} i \left| P(L_x(K) = i) - P(L_{x'}(K) = i) \right|, \end{aligned}$$

According to the proof of theorem 5.1, we have for a certain probability $p < 1$, that $P(L_y(K) = i) \leq p^i$ for all $y \in [0, K]$. Choosing an integer N such that

$$\sum_{i=N+1}^{\infty} i p^i < \frac{\epsilon}{4},$$

we find that we only have to consider a series with a finite number of terms:

$$\left| E(L_K; x) - E(L_K; x') \right| < \frac{\epsilon}{2} + \sum_{i=1}^N i \left| P(L_x(K) = i) - P(L_{x'}(K) = i) \right|.$$

But we can derive a bound for the terms of that series:

$$\begin{aligned} & \left| P(L_x(K) = i) - P(L_{x'}(K) = i) \right| = \\ & \left| P(L_x(K) = i, L_{x'}(K) \neq i) - P(L_{x'}(K) = i, L_x(K) \neq i) \right| \leq \\ & 2 P(L_x(K) \neq L_{x'}(K)). \end{aligned}$$

It is clear that $P(L_x(K) \neq L_{x'}(K)) < \epsilon/2N(N+1)$ yields the stated. This can be achieved as follows. It is clear that a necessary condition for the event $\{L_x(K) \neq L_{x'}(K)\}$ is that there exists a work requirement such that starting in x there would be no overflow, whereas starting in x' level K is reached. For the first work requirement with this property, say B_n , it holds obviously that B_n has to be in an interval of length $x - x'$. Now it follows that

$$\begin{aligned} P(L_x(K) \neq L_{x'}(K)) &= \sum_{j=0}^{\infty} P(L_x(K) \neq L_{x'}(K) | L_{x'}(K) = j) \times P(L_{x'}(K) = j) \\ &\leq \sum_{j=0}^{\infty} P(L_x(K) \neq L_{x'}(K) | L_{x'}(K) = j) \times p^j \\ &\leq \sum_{j=0}^{\infty} j \sup_{a \geq 0} P(a \leq B_1 < a + (x - x')) \times p^j \\ &= \frac{p}{1 - p^2} \times \sup_{a \geq 0} P(a \leq B_1 < a + (x - x')). \end{aligned}$$

Note that the second inequality is justified by the subadditivity of the probability measure. Choosing x' such that

$$\frac{p}{1 - p^2} \times \sup_{a \geq 0} P(a \leq B_1 < a + (x - x')) < \frac{\epsilon}{2N(N+1)},$$

we find the desired result

$$\left| E(L_K; x) - E(L_K; x') \right| < \frac{\epsilon}{2} + \sum_{i=1}^N 2i \frac{\epsilon}{2N(N+1)} = \epsilon.$$

In view of the arbitrariness of the choice of ϵ , we obtained that the investigated function is continuous. ■

Also its limit function for $K \rightarrow \infty$, $E(L_\infty; x)$, can be proven to be continuous. Since the bounds used in lemma 5.2 actually do not depend on buffer size K , we can simply copy the proof.

▷LEMMA 5.3

$E(L_\infty; x)$ is a continuous function of x , where $x \in [0, \infty)$.

The next useful analytical lemma can be found in several mathematical textbooks, for instance Rudin [1964, page 150]. It gives some sufficient conditions for *uniform convergence*. We state the lemma without proof.

▷LEMMA 5.4

Let C be compact. Suppose

- $(f_n)_{n>0}$ are continuous functions on C ;
- $(f_n)_{n>0}$ converges pointwise to a continuous function f on C ;
- for fixed $x \in C$, $f_n(x)$ decreases in n .

Then $f_n \rightarrow f$, uniformly on C . In other words:

$$\sup_{x \in C} |f_n(x) - f(x)| \rightarrow 0 \quad (n \rightarrow \infty).$$

Using lemmas 5.2, 5.3, and 5.4, we are able to prove that $\lim_{K \rightarrow \infty} E(L_K^{(2)} | \text{loss cycle})$ exists. The triangle inequality yields

$$\begin{aligned} & \left| E(L_\infty^{(2)} | \text{loss cycle}) - E(L_K^{(2)} | \text{loss cycle}) \right| = \\ & \left| \int_0^\infty E(L_\infty; x) dP(x) - \int_0^K E(L_K; x) dP_K(x) \right| \leq \\ & \left| \int_0^\infty E(L_\infty; x) dP(x) - \int_0^K E(L_\infty; x) dP_K(x) \right| + \\ & \left| \int_0^K E(L_\infty; x) dP_K(x) - \int_0^K E(L_K; x) dP_K(x) \right|. \end{aligned}$$

We prove that both last terms converge to 0.

- First consider

$$\left| \int_0^\infty E(L_\infty; x) dP(x) - \int_0^K E(L_\infty; x) dP_K(x) \right|.$$

Since $E(L_\infty; x)$ is continuous and bounded and $P_K(x) \rightarrow P(x)$ (where $x \in [0, \infty)$), we see that, using convergence in distribution, this term converges to 0.

- Finally, we investigate

$$\left| \int_0^K E(L_\infty; x) dP_K(x) - \int_0^K E(L_K; x) dP_K(x) \right|.$$

It is clear that $E(L_\infty; x) \rightarrow 0$ ($x \rightarrow \infty$). This can be proved as follows

$$\begin{aligned}
0 &\leq \limsup_{x \rightarrow \infty} E(L_\infty; x) \\
&\leq \limsup_{x \rightarrow \infty} \sum_{i=0}^{\infty} i \limsup_{K \rightarrow \infty} P(L_x(K) = i) \\
&\leq \limsup_{x \rightarrow \infty} \sum_{i=0}^{\infty} \beta(x) i p^{i-1} = \frac{1}{(1-p)^2} \lim_{x \rightarrow \infty} \beta(x) = 0,
\end{aligned}$$

according to lemma 3.6. Obviously, also $\liminf_{x \rightarrow \infty} E(L_\infty; x) = 0$, which implies that $E(L_\infty; x) \rightarrow 0$ when $x \rightarrow \infty$. Here p is defined as in theorem 5.1.

Thus, there exists an x_0 such that for all $x \geq x_0$ we have that $E(L_\infty; x)$ is smaller than $\epsilon/2$. If K runs from x_0 to ∞ , we see

$$\begin{aligned}
&\left| \int_0^K E(L_\infty; x) dP_K(x) - \int_0^K E(L_K; x) dP_K(x) \right| \leq \\
&\quad \left| \int_{[0, x_0]} (E(L_\infty; x) - E(L_K; x)) dP_K(x) \right| + \\
&\quad \left| \int_{(x_0, K)} (E(L_\infty; x) - E(L_K; x)) dP_K(x) \right|.
\end{aligned}$$

Because of the choice of x_0 ,

$$\left| \int_{(x_0, K)} (E(L_\infty; x) - E(L_K; x)) dP_K(x) \right| < \frac{\epsilon}{2}.$$

Now we still have to prove that there exists a K_0 such that for all K larger than K_0

$$\left| \int_{[0, x_0]} (E(L_\infty; x) - E(L_K; x)) dP_K(x) \right| < \frac{\epsilon}{2}.$$

We invoke lemma 5.4 (with respect to the compactum $[0, x_0]$). The first condition is satisfied because of lemma 5.2; the second because of lemma 5.3; the third since $E(L_\infty; x) - E(L_K; x)$ decreases in K for fixed $x \in [0, x_0]$. Applying lemma 5.4, there exists a K_0 such that for all $K \geq K_0$

$$\sup_{x \in [0, x_0]} |E(L_\infty; x) - E(L_K; x)| < \frac{\epsilon}{2},$$

which implies the stated.

▷ THEOREM 5.5

The ratio of $\alpha(K)$ and $\pi_i(K)$ converges to a certain constant, when $K \rightarrow \infty$ ($i = 1, 2$).

We may conclude that, provided that the distributions of $A_1 - A_0$ and B_1 are such that there exists a θ with the desired property, we proved our second objective. As mentioned before, θ denotes the unique positive solution of $M_{A_1 - A_0}(-\sigma\theta)M_{B_1}(\theta) = 1$. Supposing

that the customers arrive in accordance with a Poisson process with rate λ , this equation can be simplified as follows. Using integration by parts,

$$\begin{aligned} M_{A_1-A_0}(-\sigma\theta)M_{B_1}(\theta) &= \frac{\lambda}{\lambda+\sigma\theta} \int_0^\infty e^{\theta y} dB(y) \\ &= \frac{\lambda}{\lambda+\sigma\theta} \left(1 + \int_0^\infty \theta e^{\theta y} (1-B(y)) dy \right). \end{aligned}$$

Therefore, $M_{A_1-A_0}(-\sigma\theta)M_{B_1}(\theta) = 1$ is equivalent to $\theta = 0$ or

$$\frac{\lambda}{\sigma} \int_0^\infty e^{\theta y} (1-B(y)) dy = 1.$$

We proved the conjecture given in Tijms [1986, page 330] for the long-run fraction of rejected customers in the model with a Poissonian arrival pattern and complete rejection.

6 Estimation of $\pi_i(K)$ by fast simulation

IN THE PREVIOUS sections we proved the following asymptotical result for the long-run fraction of customers causing an overflow:

$$\pi_i(K) \approx \zeta_i e^{-\theta K}$$

for large K and $i = 1, 2$. Whereas we can solve θ , for the amplitude factors ζ_1 and ζ_2 we did not find an explicit expression. We know that ζ_1 and ζ_2 are constants, but we do not know their values. By means of simulation, we may try to estimate them. Moreover, the approximation above is only valid for large values of K , so for small K simulation is an important tool to obtain information about the values of $\pi_1(K)$ and $\pi_2(K)$.

To estimate the $\pi_i(K)$ ($i = 1, 2$) we simulate the (complete or partial rejection) queueing process. Supposing that n customers entered the system, an estimate of $\pi_i(K)$ ($i = 1, 2$) is the usual *Monte Carlo estimate*: the number of customers causing an overflow divided by n .

However, the long-run fraction of customers that cause an overflow is typically small, say 10^{-9} . Clearly an enormous number of arrivals must occur to obtain an accurate estimate, which causes large simulation times. Apart from that, perhaps more random numbers are needed than the period of the random generator! This may cause unstable estimates. In the sequel of this section we present a method to cope with these problems.

In the further analysis, we will apply the following equation from section 5, where $i = 1, 2$:

$$\pi_i(K) = \frac{E(L_K^{(i)})}{E(A_K^{(i)})} = \frac{E(L_K^{(i)} \mid \text{loss cycle})}{E(A_K^{(i)})} \alpha(K).$$

It is clear that we can obtain an estimate of the long run fraction of rejected customers by estimating subsequently $E(L_K^{(i)} \mid \text{loss cycle})$, $E(A_K^{(i)})$ and $\alpha(K)$. Note that we have to

deal with the same kind of problems when estimating $\alpha(K)$ instead of $\pi_i(K)$ ($i = 1, 2$). However, a lot of time can be saved and instability can be avoided estimating $\alpha(K)$ by Importance Sampling, as we will explain now. Basic references in the field of variance reduction by Importance Sampling in conjunction with Large Deviations are Cotrell *et al.* [1983], Parekh and Walrand [1989], and Bucklew [1990].

As usual, define $\xi_i := B_i - \sigma(A_{i+1} - A_i)$, where $i \in \mathbb{N}$. We assume that the equation $M_{\xi_1}(\theta) = 1$ has exactly one positive root, which we will call simply θ . We define by $Q_{\xi_1}(\cdot)$ the exponential tilted or twisted version of $P_{\xi_1}(\cdot)$:

$$dQ_{\xi_1}(x) := e^{\theta x} dP_{\xi_1}(x),$$

The choice of θ yields that $Q_{\xi_1}(\cdot)$ is a probability measure as well. But where the mean under $P_{\xi_1}(\cdot)$ is negative

$$\int_{-\infty}^{\infty} x dP_{\xi_1}(x) = E(\xi_1) < 0,$$

conversely it holds that

$$\int_{-\infty}^{\infty} x dQ_{\xi_1}(x) = E(\xi_1 e^{\theta \xi_1}) > 0.$$

(Cf. the proof of lemma 3.2. $E(\xi_1 e^{\theta \xi_1})$ is the slope of the moment generating function $M_{\xi_1}(\cdot)$ in the point $(\theta, 1)$, which is obviously positive.) So, sampling the ξ_i not from $P_{\xi_1}(\cdot)$ but from distribution $Q_{\xi_1}(\cdot)$, the queueing process gets a positive instead of a negative drift. In other words: we are dealing with an unstable instead of a stable queueing system.

Changing the distribution of ξ_1 has several advantages. Suppose that we simulate the queueing system until $[K, \infty)$ or $(-\infty, 0]$ is hit, and we repeat this procedure several times. It turns out that K is crossed often under $Q_{\xi_1}(\cdot)$, in contrast with the case that the ξ_i are sampled from their original distribution. We see that under the new distribution rare events occur more frequently. Furthermore, the drift of the new process (i.e., a straight line with slope $E(\xi_1 e^{\theta \xi_1})$) equals the optimum trajectory to a high level, as mentioned in section 2. It can be seen that in some sense this change of measure is optimal (Cotrell *et al.* [1983], page 910). Roughly speaking, if the buffer contents reaches a high level, it is as if the ξ_i are sampled from $Q_{\xi_1}(\cdot)$, see also the final paragraph of section 3.

To simulate the queueing model under the new distribution, we have to know the new densities of the amount of work brought along and the amount of work processed between two arrivals. We assume that $-\sigma(A_1 - A_0)$ and B_1 have densities $a(\cdot)$ and $b(\cdot)$, respectively. Trivially we have

$$dP_{\xi_1}(x) = \int_{-\infty}^{\infty} b(x - y) a(y) dy,$$

which results in the new density

$$dQ_{\xi_1}(x) = \int_{-\infty}^{\infty} e^{\theta x} b(x - y) a(y) dy = \int_{-\infty}^{\infty} e^{\theta(x-y)} b(x - y) e^{\theta y} a(y) dy =$$

$$\int_{-\infty}^{\infty} \frac{e^{\theta(x-y)} b(x-y)}{M_{B_1}(\theta)} \frac{e^{\theta y} a(y)}{M_{A_1-A_0}(-\sigma\theta)} dy$$

Under distribution $Q_{\xi_1}(\cdot)$ we see that ξ_1 is distributed as the sum of two random variables having densities

$$b^*(x) := \frac{e^{\theta x} b(x)}{\int_0^{\infty} e^{\theta y} b(y) dy} = \frac{e^{\theta x} b(x)}{M_{B_1}(\theta)}$$

for non-negative x and

$$a^*(x) := \frac{e^{\theta x} a(x)}{\int_0^{\infty} e^{-\sigma\theta y} a(y) dy} = \frac{e^{\theta x} a(x)}{M_{A_0-A_1}(-\sigma\theta)}$$

for $x \leq 0$. We find that under the twisted distribution we may let B_1 have density $b^*(\cdot)$ and $-\sigma(A_1 - A_0)$ density $a^*(\cdot)$.

Simulate the queueing process under $Q_{\xi_1}(\cdot)$ until either a value in $[K, \infty)$ or in $(-\infty, 0]$ is attained. We will call this a run. Perform n_x runs. It is clear that estimating $\alpha(K)$ by the number of runs in which the interval $[K, \infty)$ is reached divided by n_x would give too large values. Obviously, we have to measure how more likely K is exceeded in a run under $Q_{\xi_1}(\cdot)$ than under $P_{\xi_1}(\cdot)$ to compensate the fact that reaching K occurs more frequently under the twisted distribution. Therefore, we determine likelihood X_j of run j . Suppose that in run j the interval $[K, \infty)$ is hit via the path $(\omega_0, \dots, \omega_m)$, where $\omega_0 = 0$, $\omega_m \geq K$ and $\omega_i \in (0, K)$ for $i \in \{1, \dots, m-1\}$. Then the likelihood ratio (cf. *Radon-Nikodým derivative*) of this run is

$$x_j = \prod_{i=1, \text{ odd}}^m a_i^{(o)} \prod_{i=2, \text{ even}}^{m-1} a_i^{(e)},$$

Here $a_i^{(o)}$ is defined as the likelihood ratio of a step upwards from ω_{i-1} to ω_i , which reduces to

$$a_i^{(o)} = \frac{M_{B_1}(\theta)}{e^{\theta(\omega_i - \omega_{i-1})}}.$$

On the other hand $a_i^{(e)}$ is the likelihood ratio of a step downwards from ω_{i-1} to ω_i :

$$a_i^{(e)} = \frac{M_{A_1-A_0}(-\sigma\theta)}{e^{\theta(\omega_i - \omega_{i-1})}}.$$

It turns out that $x_j = M_{B_1}(\theta) \exp(-\theta\omega_m)$. Put $x_j := 0$ if in run j the interval $(-\infty, 0]$ is hit. It can be proved that $x_j < 1$ (for $j \in \{1, \dots, n_x\}$) guarantees that we get a more accurate estimate using the sample mean

$$\bar{x}(n_x) := \sum_{j=1}^{n_x} x_j / n_x$$

instead of the estimate obtained by simulating under $P_{\xi_1}(\cdot)$, see Walrand [1988, pages 336-337]. It can be proved $\bar{X}(n_x)$ converges to $E(X_1) = \alpha(K)$ almost surely ($n_x \rightarrow \infty$). Note that $x_j < 1$ is achieved for

$$K > \log(M_{B_1}(\theta)) / \theta.$$

If cycle j is a loss cycle, we have a good starting point for the simulation of the number of customers lost in a loss cycle Y_j . In case of partial rejection, this starting point is obviously K . On the contrary dealing with complete rejection, we start to simulate in ω_{m-1} . (This is justified by the fact that, if a high level is reached by the queueing process, the ξ_i seem to be sampled from distribution $Q_{\xi_1}(\cdot)$.) The simulation of Y_j should be performed under $P_{\xi_1}(\cdot)$. The realization of this simulation is denoted by y_j , where $j \in \{1, \dots, n_y\}$. An unbiased estimate for $E(L_K^{(i)} \mid \text{loss cycle})$ is

$$\bar{y}(n_y) := \sum_{j=1}^{n_y} y_j / n_y.$$

According to the strong law of large numbers $\bar{Y}(n_y) \rightarrow E(Y_1)$ with probability 1, $n_y \rightarrow \infty$.

Finally, simulate n_z cycles of the (partial or complete rejection) queueing distribution under distribution P_{ξ_1} . Let Z_j denote the number of arrivals in cycle j , z_j its realization. Again, an unbiased estimate for $E(A_K^{(i)})$ is its sample mean $\bar{z}(n_z)$. The estimator $\bar{Z}(n_z)$ also converges almost surely to $E(Z_1)$ if n_z tends to ∞ . An estimator for $\pi_i(K)$ is

$$E(n_x, n_y, n_z) := \frac{\bar{X}(n_x) \bar{Y}(n_y)}{\bar{Z}(n_z)}.$$

Since almost sure convergence survives continuous transformations, we see that this estimator converges to $\pi_i(K)$ with probability 1, if $n_x, n_y, n_z \rightarrow \infty$.

However, we do not want only an point estimator for $\pi_i(K)$, we also want to know how ‘good’ the obtained estimate is. Estimating the ratio of two means, it is possible to construct a confidence interval using the central limit theorem, see for instance Law and Kelton [1986, pages 299-300]. Unfortunately, a similar approach seems to be impossible in this case. Therefore, we have to use other techniques to do statements about the accuracy of the estimate of the quantity of interest. We introduce the notion *Level of Confidence* (see also Walrand [1988, page 335]).

▷ DEFINITION 6.1

An estimate $\hat{\alpha}$ of α has Level of Confidence $(a, b) \in (0, 1) \times (0, 1)$ if

$$P \left(\left| \frac{\hat{\alpha} - \alpha}{\alpha} \right| \leq a \right) \geq b.$$

The parameter a is called the relative precision; b denotes the confidence. Usual values are for instance $a = 0.2$ and $b = 0.9$ (i.e., we can claim with 90% confidence that the true value is contained in the confidence interval; the ratio of the confidence interval half-length to the true value is smaller than 20%.)

Now we try to find a lower bound for the probability

$$P \left(\left| \frac{E(n_X, n_Y, n_Z) - \pi_i(K)}{\pi_i(K)} \right| \leq a \right)$$

to see what confidence is guaranteed given relative precision a . First we define $a_+ := \sqrt[3]{1+a}$ and $a_- := \sqrt[3]{1-a}$, and the following intervals:

$$\begin{aligned} I_x &:= [E(X_1)a_-, E(X_1)a_+] & I'_x &:= [E(X_1)(2-a_+), E(X_1)a_+] \\ I_y &:= [E(Y_1)a_-, E(Y_1)a_+] & I'_y &:= [E(Y_1)(2-a_+), E(Y_1)a_+] \\ I_z &:= [E(Z_1)/a_+, E(Z_1)/a_-] & I'_z &:= [E(Y_1)/a_+, E(Y_1)(2-1/a_+)] \end{aligned}$$

It is easy to derive that the intervals I'_x, I'_y , and I'_z are symmetric with respect to means $E(X_1), E(Y_1)$ and $E(Z_1)$, respectively. Now consider the following trivial inclusions:

$$\begin{aligned} \left\{ \left| \frac{E(n_x, n_y, n_z) - \pi_i(K)}{\pi_i(K)} \right| \leq a \right\} &\supseteq \\ \{ \bar{X}(n_x) \in I_x \cap \bar{Y}(n_y) \in I_y \cap \bar{Z}(n_z) \in I_z \} &\supseteq \\ \{ \bar{X}(n_x) \in I'_x \cap \bar{Y}(n_y) \in I'_y \cap \bar{Z}(n_z) \in I'_z \} \end{aligned}$$

Noting that $\{\bar{X} \notin I'_x\}$ implies that $|\bar{X} - E(\bar{X}(n_x))| > E(X_1)(a_+ - 1)$, we find the following upper bound, based on Chebyshev's inequality (for instance Williams [1991, page 73])

$$P(\bar{X}(n_x) \notin I'_x) \leq \frac{\text{Var}(\bar{X}(n_x))}{E^2(X_1)(a_+ - 1)^2},$$

and an analogous expression for $\bar{Y}(n_y)$. For $\bar{Z}(n_z)$ we find

$$P(\bar{Z}(n_z) \notin I'_z) \leq \frac{\text{Var}(\bar{Z}(n_z))}{E^2(Z_1)(1 - 1/a_+)^2}.$$

Note that in case of complete rejection $\{X_j, j = 1, \dots, n_x\}$ and $\{Y_j, j = 1, \dots, n_y\}$ are 'almost' independent sequences. This can be seen as follows. Suppose that in cycle j level K is reached by trajectory $(\omega_0, \dots, \omega_m)$ then x_j is determined by $(\omega_0, \dots, \omega_m)$, whereas the appropriate realization of the number of customers lost in a loss cycle mainly depends on $(\omega_{m+1}, \omega_{m+2}, \dots)$. In case of partial rejection, both sequences are obviously completely independent.

The observations above yield

$$\begin{aligned} P\left(\left|\frac{E(n_x, n_y, n_z) - \pi_i(K)}{\pi_i(K)}\right| \leq a\right) &\geq \\ \left(1 - \frac{\text{Var}(X_1)/n_x}{E^2(X_1)(a_+ - 1)^2}\right) &\left(1 - \frac{\text{Var}(Y_1)/n_y}{E^2(Y_1)(a_+ - 1)^2}\right) \left(1 - \frac{\text{Var}(Z_1)/n_z}{E^2(Z_1)(1 - 1/a_+)^2}\right). \end{aligned}$$

We may estimate $\text{Var}(X_1)$ by the sample variance

$$\frac{\sum_{j=1}^{n_x} (x_j - \bar{x}(n_x))^2}{n_x - 1}$$

($\text{Var}(Y_1)$ and $\text{Var}(Z_1)$ analogously). Sample mean $\bar{x}(n_x)$ can be used as estimate for $E(\xi_1)$ ($E(Y_1)$ and $E(Z_1)$ analogously). Deciding that the sequences $\{X_j, j = 1, \dots, n_x\}$ and $\{Y_j, j = 1, \dots, n_y\}$ are not independent, according to the Bonferroni inequality (Law and Kelton [1986, page 308]) we still have the lower bound

$$1 - \frac{\text{Var}(X_1)/n_x}{E^2(X_1)(a_+ - 1)^2} - \frac{\text{Var}(Y_1)/n_y}{E^2(Y_1)(a_+ - 1)^2} - \frac{\text{Var}(Z_1)/n_z}{E^2(Z_1)(1 - 1/a_+)^2},$$

which is a bit weaker than the bound above. Noting that $a_+ \approx 1 + \frac{1}{3}a$ and $1/a_+ \approx 1 - \frac{1}{3}a$, the bounds obtained can be simplified to

$$1 - \frac{9}{a^2} \left(\frac{\text{Var}(X_1)/n_x}{E^2(X_1)} + \frac{\text{Var}(Y_1)/n_y}{E^2(Y_1)} + \frac{\text{Var}(Z_1)/n_z}{E^2(Z_1)} \right).$$

Given relative precision a , we can find in this way a guaranteed confidence b . But, since the used inequalities may be not sharp, the actual Level of Confidence is probably considerable higher than this guaranteed Level of Confidence (a, b) . Note that relative precision and confidence are in some sense exchangeable: a higher accuracy a yields a lower confidence b .

To conclude this subsection, we give some simulation results. In the case of Poisson arrivals (i.e., the interarrival times are exponentially distributed, say with mean λ^{-1}) excellent approximations are available for the buffer size K such that $\pi_i(K)$ is smaller than a given fraction ν , where $\nu \in (0, 1)$. We call this buffer size $K_i(\nu)$, $i = 1, 2$. In this way, we can compare ν with the simulation estimate. In the remainder, we take for the sake of convenience $E(B_1) = 1$ and $\sigma = 1$. Furthermore, we choose the parameters of the distribution of work requirement B_1 such that its squared coefficient of variation (Tijms [1986, page 393]), say c_B^2 , equals $\frac{1}{2}$.

We let $\theta^{(\text{det})}$, $\zeta_1^{(\text{det})}$ and $\zeta_2^{(\text{det})}$, respectively $\theta^{(\text{exp})}$, $\zeta_1^{(\text{exp})}$ and $\zeta_2^{(\text{exp})}$ be the corresponding values of θ , ζ_1 and ζ_2 in case of B_1 having a deterministic and exponential distribution, respectively. In for itself speaking notation, it can be found in Tijms [1986, pages 314 and 329] that the following two-moment approximation is valid:

$$\begin{aligned} K_i(\nu) &\approx c_B^2 K_i^{(\text{exp})}(\nu) + (1 - c_B^2) K_i^{(\text{det})}(\nu) \\ &\approx \log \left(\frac{\zeta_i^{(\text{exp})}}{\nu} \right) / 2\theta^{(\text{exp})} + \log \left(\frac{\zeta_i^{(\text{det})}}{\nu} \right) / 2\theta^{(\text{det})}, \end{aligned}$$

where $i = 1, 2$. Denoting the offered load $\lambda E(B_1)/\sigma$ by ρ , we have according to Tijms [1986, pages 57-59, 311, and 326-329],

$$\theta^{(\text{det})} = \frac{1}{E(B_1)} \log \left(1 + \frac{\sigma \theta^{(\text{det})}}{\lambda} \right) = \log \left(1 + \frac{\theta^{(\text{det})}}{\lambda} \right),$$

$$\begin{aligned}
\zeta_1^{(\text{det})} &= \frac{(1-\rho)\sigma\theta^{(\text{det})}}{\lambda(\theta^{(\text{det})}E(B_1) - (1-\rho))} = \frac{(1-\rho)\theta^{(\text{det})}}{\lambda(\theta^{(\text{det})} - (1-\rho))}, \\
\zeta_2^{(\text{det})} &= \frac{(1-\rho)^2 \exp(\theta^{(\text{det})}E(B_1))}{\theta^{(\text{det})}E(B_1) - (1-\rho)} = \frac{(1-\rho)^2 \exp(\theta^{(\text{det})})}{\theta^{(\text{det})} - (1-\rho)}, \\
\theta^{(\text{exp})} &= \frac{1-\rho}{E(B_1)} = 1-\rho, \\
\zeta_1^{(\text{exp})} &= \frac{\sigma\theta^{(\text{exp})}\rho}{\lambda} = \theta^{(\text{exp})}, \\
\zeta_2^{(\text{exp})} &= (1-\rho)e^{-\rho}.
\end{aligned}$$

Note that the first equation of this display can be solved numerically, using a fixed point iteration scheme.

For the work the customer brought along, we focus on three distributions. These distributions have two parameters each to make sure that we can choose them such that $E(B_1) = 1$ and $c_B^2 = \frac{1}{2}$. First we consider the Erlang(2) distribution with shape parameter μ . Since $E(B_1) = 1$, we see immediately that $\mu = 2$. Denoting by θ the solution of $M_{\xi_1}(\theta) = 1$, we obtain that the twisted version of the distribution of B_1 is again Erlang(2), but with shape parameter $\mu - \theta$.

Then B_1 is distributed as a mixture of Erlang(1) and Erlang(3) random variables with the same shape parameter μ . Under the new distribution, B_1 is still a mixture of Erlang(1) and Erlang(3) distributions, but the weights have changed and the shape parameters become $\mu - \theta$.

Finally, B_1 has a shifted exponential density : $B_1 \stackrel{d}{=} E + l$, for some positive constant l and E distributed exponentially with mean μ^{-1} . It can be seen that the twisted version of this distribution is again shifted exponential with location parameter l , but with decay rate $\mu - \theta$.

Noting that the new distribution of the interarrival times is exponential with mean $(\lambda + \theta)^{-1}$, we are ready to implement the fast simulation of the queueing systems. In the next tables, we choose the buffer size such that the long-run fraction of customers that cause an overflow is about 10^{-3} , 10^{-6} and 10^{-9} , using the two-moment approximation given above. Fixing the relative precision at 20%, we get the following results.

Partial rejection

$\nu = 10^{-3}$	Erlang(2)	Mixed Erlang(1,3)	Shifted Exp
$\rho = 0.2, K = 5.80$	$2.06 \cdot 10^{-3}$ (96%)	$1.00 \cdot 10^{-3}$ (96%)	$2.79 \cdot 10^{-3}$ (94%)
$\rho = 0.5, K = 9.16$	$1.61 \cdot 10^{-3}$ (96%)	$1.18 \cdot 10^{-3}$ (96%)	$1.71 \cdot 10^{-3}$ (97%)
$\rho = 0.8, K = 20.39$	$1.31 \cdot 10^{-3}$ (92%)	$1.28 \cdot 10^{-3}$ (90%)	$1.32 \cdot 10^{-3}$ (91%)

Complete rejection

$\nu = 10^{-3}$	Erlang(2)	Mixed Erlang(1,3)	Shifted Exp
$\rho = 0.2, K = 5.65$	$1.41 \cdot 10^{-3}$ (94%)	$0.92 \cdot 10^{-3}$ (95%)	$1.98 \cdot 10^{-3}$ (94%)
$\rho = 0.5, K = 8.53$	$1.25 \cdot 10^{-3}$ (96%)	$1.04 \cdot 10^{-3}$ (96%)	$1.54 \cdot 10^{-3}$ (97%)
$\rho = 0.8, K = 17.76$	$1.24 \cdot 10^{-3}$ (92%)	$1.08 \cdot 10^{-3}$ (93%)	$1.26 \cdot 10^{-3}$ (93%)

Partial rejection

$\nu = 10^{-6}$	Erlang(2)	Mixed Erlang(1,3)	Shifted Exp
$\rho = 0.2, K = 11.42$	$1.72 \cdot 10^{-6}$ (96%)	$0.85 \cdot 10^{-6}$ (96%)	$5.23 \cdot 10^{-6}$ (97%)
$\rho = 0.5, K = 18.82$	$1.63 \cdot 10^{-6}$ (94%)	$1.12 \cdot 10^{-6}$ (94%)	$2.24 \cdot 10^{-6}$ (97%)
$\rho = 0.8, K = 45.69$	$1.23 \cdot 10^{-6}$ (91%)	$1.11 \cdot 10^{-6}$ (91%)	$1.48 \cdot 10^{-6}$ (90%)

Complete rejection

$\nu = 10^{-6}$	Erlang(2)	Mixed Erlang(1,3)	Shifted Exp
$\rho = 0.2, K = 11.27$	$1.40 \cdot 10^{-6}$ (94%)	$0.79 \cdot 10^{-6}$ (95%)	$2.32 \cdot 10^{-6}$ (94%)
$\rho = 0.5, K = 18.18$	$1.37 \cdot 10^{-6}$ (96%)	$1.07 \cdot 10^{-6}$ (96%)	$1.94 \cdot 10^{-6}$ (94%)
$\rho = 0.8, K = 43.04$	$1.35 \cdot 10^{-6}$ (92%)	$1.25 \cdot 10^{-6}$ (93%)	$1.40 \cdot 10^{-6}$ (90%)

Partial rejection

$\nu = 10^{-9}$	Erlang(2)	Mixed Erlang(1,3)	Shifted Exp
$\rho = 0.2, K = 17.04$	$1.46 \cdot 10^{-9}$ (96%)	$0.56 \cdot 10^{-9}$ (95%)	$9.67 \cdot 10^{-9}$ (96%)
$\rho = 0.5, K = 28.48$	$1.49 \cdot 10^{-9}$ (96%)	$0.95 \cdot 10^{-9}$ (94%)	$3.55 \cdot 10^{-9}$ (97%)
$\rho = 0.8, K = 70.99$	$1.27 \cdot 10^{-9}$ (90%)	$1.12 \cdot 10^{-9}$ (91%)	$1.66 \cdot 10^{-9}$ (91%)

Complete rejection

$\nu = 10^{-9}$	Erlang(2)	Mixed Erlang(1,3)	Shifted Exp
$\rho = 0.2, K = 16.88$	$1.01 \cdot 10^{-9}$ (95%)	$0.55 \cdot 10^{-9}$ (95%)	$6.23 \cdot 10^{-9}$ (97%)
$\rho = 0.5, K = 27.84$	$1.32 \cdot 10^{-9}$ (96%)	$0.87 \cdot 10^{-9}$ (94%)	$2.40 \cdot 10^{-9}$ (93%)
$\rho = 0.8, K = 68.35$	$1.28 \cdot 10^{-9}$ (92%)	$1.16 \cdot 10^{-9}$ (91%)	$1.57 \cdot 10^{-9}$ (91%)

We added the guaranteed confidence between parentheses. We chose n_x and n_z such that this confidence was at least 90%. In the complete rejection case, the simulation results can be compared with those in Tijms [1986, page 330]. In case of partial rejection, buffersize $K_1(\nu)$ differs slightly from the values found in Tijms [1986, page 315]. This is a consequence of the fact that we did not use exact values for $K_1^{(\text{det})}(\nu)$ and $K_1^{(\text{exp})}(\nu)$.

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