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ERROR BOUNDS FOR ARBITRARY APPROXIMATIONS
OF "NEARLY REVERSIBLE" MARKOV CHAINS
AND AN ALOHA-APPLICATION

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Abstract A condition is provided to conclude error bounds for an arbitrary steady state approximation of a "nearly reversible" Markov chain. The error bound is of the form $\Delta \cdot R$, where

- (i) Δ can be computed merely by the approximation in order,
- (ii) R is to be obtained by bounding so-called bias terms for the system of interest. This can be established analytically.

The results will be illustrated for ALOHA-type systems with different source characteristics. An approximation is suggested based on truncating the corresponding Möbius-function. The R -value is computed by an inductive Markov reward proof technique. Numerical illustration indicates that the analytic error bound can be reasonable for practical purposes.

Keywords Markov chain * Approximation * Error bound * Nearly reversible * Bias terms * ALOHA-systems.

1. Introduction

Motivation

Markov obtain theory has proven to be a powerful tool for performance evaluation of computer and communication systems. Unfortunately, though, such systems rarely exhibit a closed form expression, such as most notably a product form, due to practical phenomena as blocking or interferences of different sources. As exact numerical analysis is computationally most expensive or even infeasible, approximate analysis are widely employed.

Approximative approaches are usually supported by extensive experimental illustration and heuristic or intuitive argumentation. Analytic *a priori* or on line error bounds, however, are rarely reported and seem more or less restricted to numerical or exact decomposition and aggregation procedures (cf. [4], [5], [10], [11], [13], [15], [20]).

Robust but secure error bounds are of practical interest to obtain:

- (i) A 100% secure order of magnitude.
- (ii) A restricted interval of possible values to which attention can be restricted, such as for simulation or optimization purposes.
- (iii) A guarantee of possible correctness or incorrectness of model assumptions, conjectures or approximate approaches.

Recently, in [16] and [17] conditions have been provided to conclude error bounds for the effect of possible parameter perturbations and/or state space truncations. Various approximations though are not simply minor perturbations and/or truncations but may involve a totally different underlying law of motion or even not be interpretable as corresponding to some modified system.

For example, approximations for queueing networks with blocking and/or failures are usually based on (iteratively) adapting effective service rates as if service stations can be regarded in isolation and using these in the analogue system without such features (see for example various papers in [1]). Or approximations may follow by analytic simplifications which do no longer fit a direct probabilistic or system descriptive interpretation (cf. [19]).

General result

This paper will provide a tool to conclude error bounds for approximate results based on any approximation for the steady state distribution. These error bounds can be expected to be "reasonable" when the system is "nearly reversible".

Nearly reversible

Reversibility (cf. [8]) is a most-important property in queueing network theory, on the one hand as it can be shown to be an indirect characterization of product form results (cf. [7], [8]) (which does not mean that the system itself has to be reversible), while on the other hand it leads to simple computations. Though strict reversibility of communication networks is limited to rather special queueing networks under simplifying assumptions as no collisions, no propagation delays and no retransmission, "nearly reversibility" seems quite common in practical communications or queueing. Here "nearly reversibility" is not a standard or well-defined concept in the literature but roughly stands for strict reversibility up to some minor modification of one or a few of the underlying descriptions or up to some reasonably small discrepancy in the reversibility (balance) equations. For example, a single server system with breakdowns is reversible up to the rare occurrence of these breakdowns or a communication network such as an ALOHA or a CSMA system is reversible up to the occurrence of collisions.

Steps involved

Two steps are involved in order to establish an error bound of the form ΔR :

- (i) The definition of an artificial Markov based on the approximation in order from which a difference value Δ can be computed directly.
- (ii) The estimation of so-called bias terms by a value R for a given underlying Markov reward structure. This step does not depend on the used approximation and in concrete situations can usually be established by an inductive Markov reward proof technique.

By providing the estimate R , error bounds ΔR can thus be compared for various alternative approximations.

Special application

Most of the paper though will be concerned with the illustration of both steps for a particular application of practical interest. An ALOHA-system with inhomogeneous source characteristics and collision probabilities. To perform step 1 an approximation will be given based on so-called Möbius-expansions. This approximation is chosen as it cannot just be seen as some sort of physical modification (or perturbation) of the original system. A simple estimate on the essential bias-term will be derived. An explicit error bound for the system throughput is hereby obtained. Numerical support for this application indicates that the error bound can be practical.

Related Literature

The definition of this artificial Markov chain seems to be new in the present setting but is related to the splitting of linear operators in a symmetric (or selfadjoint) and antisymmetric part (also see [2] and [3]).

The comparison of the artificial chain, once defined, and the original model is closely related to a theorem that has recently been reported in [16] and [17] to establish perturbation or truncation results. However, it does not fit in either directly. The estimation of so-called bias terms by means of an inductive proof-technique has already successfully employed in a number of queueing situations. The current application to an ALOHA-system, however, is new and involves special technicalities as state-dependent collision probabilities are dealt with. The approximation provided for this application is adopted from a recently newly developed approach in [19] as based on truncation of so-called Möbius-expansions.

Recently, furthermore, in [6] an approach has been developed by estimating arbitrary Markov chains by reversible upper and lower bound modifications. This reference though merely provides rough performance bounds and not (small) error bounds for (accurate) given approximations.

2. Model and result

Consider a continuous-time Markov chain with state space $N=\{1,2,\dots\}$ and transition rates $q(i,j)$ for a transition from state i into state j . Without restrictions of generality assume that this chain is irreducible at some set S with unique stationary distribution $\{\pi(i), i \in S\}$.

Let $\{\tilde{\pi}(i), i \in \tilde{S}\}$ be any approximate probability distribution at some subset $\tilde{S} \subset S$ and define approximate transition rates at $\tilde{S}$ by

$$\tilde{q}(i,j) = \frac{1}{2} [q(i,j) + q(j,i) [\pi(j)/\pi(i)]] \quad (2.1)$$

for all $i,j \in \tilde{S}$ while $\tilde{q}(i,j)$ is defined to be equal to 0 otherwise. Then for all i,j one directly verifies the reversibility property:

$$\tilde{\pi}(i)\tilde{q}(i,j) = \tilde{\pi}(j)\tilde{q}(j,i) \quad (2.2)$$

Without restriction of generality, also assume that the approximate Markov chain with transition rates $\tilde{q}(i,j)$ as per (2.1) is irreducible at $\tilde{S}$, so that its unique stationary distribution is given by $\{\tilde{\pi}(i), i \in \tilde{S}\}$.

(Note that the global balance equations are directly verified by summing (2.2) over all j .)

Let $\bar{r}: S \rightarrow R$ be some given function, to be interpreted as a reward rate, and consider the stationary performance measures:

$$\begin{cases} \bar{g} = \sum_{i \in S} \pi(i) \bar{r}(i) \\ \tilde{g} = \sum_{i \in \tilde{S}} \tilde{\pi}(i) \bar{r}(i) \end{cases} \quad (2.3)$$

assuming that these are well-defined. As the value $\tilde{g}$ is assumed to be computable we wish to evaluate the difference $|\bar{g} - \tilde{g}|$.

First, we make the following notational conventions. An upper bar "-" symbol indicates the approximate model while an upper bar symbol "(-)" is used when the expression is to be read for both the original and approximate model. Further, for convenience assume that for some finite Q :

$$Q \geq \sup_i \sum_j \tilde{q}(i,j) \quad (2.4)$$

Then by the standard uniformization technique (cf. [14], p. 110) the distribution $\bar{\pi}(\cdot)$ is equal to the unique stationary distribution of the discrete time Markov chain with one step transition probabilities $\bar{p}(i,j)$ defined by

$$\begin{cases} \bar{p}(i,j) = \bar{q}(i,j)/Q & (j \neq i) \\ \bar{p}(i,i) = [1 - \sum_{j \neq i} \bar{q}(i,j)]/Q \end{cases} \quad (2.5)$$

Now define one-step transition operators $\{\bar{T}_t | t=0,1,2,\dots\}$ on functions f by

$$\begin{cases} \bar{T}f(i) = \sum_j \bar{p}(i,j) f(j) \\ \bar{T}_{t+1} = \bar{T} \bar{T}_t & (t \geq 0) \\ \bar{T}_0 = I \end{cases} \quad (2.6)$$

where I is the identity operator. Hence $\bar{T}_t f(i)$ represents the expected value of function f at time t under one-step transition probabilities $\bar{p}(\cdot, \cdot)$ and starting in state i at time 0. Now define functions $\bar{V}_n, n=0,1,2,\dots$ given by

$$\bar{V}_n = Q^{-1} \sum_{t=0}^{n-1} \bar{T}_t \bar{r} \quad (2.7)$$

In words that is, $\bar{V}_n(i)$ represents the expected total reward over n periods under one-step transition probabilities $\bar{p}(\cdot, \cdot)$ starting in state i at time 0 and incurring a one-step reward $r(j)$ whenever the system is in state j . Then by virtue of the uniformization technique (cf. [14], p.110) and the irreducibility assumptions, for any $\ell \in S$ we have

$$\bar{g} = \lim_{N \rightarrow \infty} \frac{Q}{N} \bar{V}_N(\ell) \quad (2.8)$$

provided this limit exists. The value $\bar{g}$ then represents the average reward per unit of time. We note here that the factor Q^{-1} and Q in (2.7) and (2.8) could be omitted but is included for convenience later on. We can now present a general comparison result between the original model and the arbitrary approximation. A more practical corollary will be concluded directly.

Theorem 2.1 Assume that for some function Φ , some initial state $l \in \tilde{S}$ and all $i \in \tilde{S}$ and $t \geq 0$:

$$\left| (r - \tilde{r})(i) + \frac{1}{2} \sum_j [q(i, j) - q(j, i)] [\tilde{\pi}(j) / \tilde{\pi}(i)] [V_t(j) - V_t(i)] \right| \leq \alpha \Phi(i) \quad (2.9)$$

and

$$\tilde{T}_t \Phi(l) \leq \beta. \quad (2.10)$$

Then

$$|g - \tilde{g}| \leq \alpha \beta. \quad (2.11)$$

Proof By virtue of (2.6) and (2.7) we have:

$$\tilde{V}_{t+1}^{(-)} = \tilde{r}^{(-)} Q^{-1} + \tilde{T}^{(-)} V_t. \quad (2.12)$$

Hence, for any n and arbitrary $l \in \tilde{S}$:

$$\begin{aligned} (\tilde{V}_n - V_n)(l) &= (\tilde{r} - r)(l) Q^{-1} + (\tilde{T} \tilde{V}_{n-1} - T V_{n-1})(l) \\ &= (\tilde{r} - r)(l) Q^{-1} + (\tilde{T} - T) V_{n-1}(l) + \tilde{T}(\tilde{V}_{n-1} - V_{n-1})(l). \end{aligned} \quad (2.13)$$

By repeating this relation for $n = N, N-1, \dots, 1$ we find:

$$(\tilde{V}_N - V_N)(l) = \sum_{t=0}^{N-1} \tilde{T}_t \left[(\tilde{r} - r) Q^{-1} + (\tilde{T} - T) V_{N-t-1} \right](l) + \tilde{T}_N (\tilde{V}_0 - V_0)(l) \quad (2.14)$$

Further, by (2.6), (2.5) and (2.1) we obtain for any i :

$$\begin{aligned} (\tilde{T} - T) V_s(i) &= \sum_j [\tilde{p}(i, j) - p(i, j)] V_s(j) \\ &= \sum_j [\tilde{p}(i, j) - p(i, j)] [V_s(j) - V_s(i)] \\ &= - \frac{1}{2} Q^{-1} \sum_j [q(i, j) - q(j, i)] [\tilde{\pi}(j) / \tilde{\pi}(i)] \left[V_s(j) - V_s(i) \right]. \end{aligned} \quad (2.15)$$

By substituting (2.15) in (2.14), noting that $\tilde{V}_0^{(-)} = 0$, taking absolute values and using that $\tilde{T}_t$ is a monotone operator, i.e. $\tilde{T}_t f(i) \leq \tilde{T}_t g(i)$ if $f(j) \leq g(j)$ for all j , we obtain from (2.9), (2.10), (2.14) and (2.15):

$$|(\tilde{V}_N - V_N)(l)| \leq \alpha Q^{-1} \sum_{t=1}^{N-1} \tilde{T}_t \Phi(l) \leq \alpha \beta N Q^{-1}. \quad (2.16)$$

Applying (2.8) completes the proof. $\square$

Remark 2.1 (Condition 2.10 and Φ) The special function Φ is included so as to allow more flexibility in satisfying (2.9). For example, one may think of Φ to be either some simple polynomial, such as $\Phi(i)=1+i$, or some indicator function for large states, such as $\Phi(i)=1$ if $i \geq M$ and 0 otherwise where M is large. Condition (2.10) then requires the expected value of this function to remain bounded over time or even to be small. More details and illustrations of this function Φ for truncation purposes can be found in [16]. As we wish to concentrate this paper on the novel aspect of providing an error bound for an arbitrary approximation, however, we aim to focus our attention to condition (2.9), where for simplicity we assume $\Phi(.)=1$.

In particular, though condition (1.9) does in principle allow to combine the approximate values $\tilde{\pi}(.)$ with the bias-terms $V_t(j)-V_t(i)$, it is more realistic and convenient that these will be analyzed separately. This is expressed by the following practical corollary, where for simplicity we assume $\tilde{r}=r$ and $\Phi(.)=1$.

Corollary 2.2 Let $\tilde{r}=r$ and assume that for some Δ and R and all $i \in \tilde{S}$ and $t \geq 0$:

$$\left| \frac{1}{2} \sum_j [q(i,j)-q(j,i)] [\tilde{\pi}(j)/\tilde{\pi}(i)] \right| \leq \Delta \quad (2.17)$$

and for all $i, j \in S$ with $q(i,j) > 0$:

$$|V_t(j)-V_t(i)| \leq R \quad (2.18)$$

Then

$$|g-\tilde{g}| \leq \Delta R. \quad (2.19)$$

Remark 2.2 (Bounded bias-terms and verification of (2.9) or (2.18)). The differences $V_t(j)-V_t(i)$ appearing in (2.9) and (2.18) are known as so-called bias-terms of the underlying Markov reward model. While $V_t(.)$ generally grows linearly in t , these differences are generally bounded uniformly in t . More precisely, when $r(.)$ is bounded, say $|r(i)| \leq B$ for all i , by simple Markov reward arguments (e.g. [17]) one proves:

$$|V_t(j)-V_t(i)| \leq 2B \min[R_{ij}, R_{ji}]$$

where R_{ij} is the expected number of steps (mean first passage-time, cf.

[9]) to reach state j out of state i . Particularly, for finite Markov chains these differences can thus be bounded uniformly in t and i, j . For the unbounded case similar though more technical results hold (cf. [..]). Unfortunately, explicit expressions or even simple bounds for mean first passage times are limited to simple one-dimensional random walks (cf. [9]). To this end, in section 4 we will illustrate how estimates for these bias-terms can be derived directly also in multi-dimensional situations by using inductive Markov reward arguments. This technique has already been applied successfully in various queueing network applications (cf. [16]). The present application is new and involves special technicalities which require an analysis in full detail.

It can also be of interest to investigate whether the proposed approximation provides an upper or lower bound of some performance measure. To this end, a more relaxed form of (2.9) can be given.

Corollary 2.3 (Comparison result) *Suppose that for all $i \in \tilde{S}$ and $t \geq 0$:*

$$(r - \tilde{r})(i) + \frac{1}{2} \sum_j [q(i, j) - q(j, i)] [\tilde{\pi}(j) / \tilde{\pi}(i)] [V_t(j) - V_t(i)] \geq (\leq) 0 \quad (2.20)$$

then

$$g \geq (\leq) \tilde{g}. \quad (2.21)$$

Proof This follows directly by substituting (2.15) and (2.20) in (2.14), recalling that the operators $\tilde{T}_t$ are monotone and applying (2.8).

□

3. Application: An ALOHA-system with inhomogeneous characteristics

This section deals with a special application in order to illustrate the results of section 2, most notably condition (2.17). Herein, for presentational convenience and clarity we restrict to an ALOHA-communication system with only four sources. The extension to any number of sources, however, merely involves more complexity but is essentially the same. The approximation used is based on a recently developed approach in [19] by truncating Möbius expansions. We choose this approximation for two reasons:

- (i) to illustrate the results for an approximation that cannot be seen as just a simple modification of the system protocols or law of motion.
- (ii) to advocate this new approach which we believe to be promising

3.1 Model

Consider a communication system with M sources (transmitters or processors) of which each can be in an idle (non-transmitting) or busy (transmitting) mode as follows. When idle a source h will schedule transmission requests at exponential times with parameter γ_h . Its transmission has an exponential time with parameter μ_h . However, with $H = \{h_1, \dots, h_n\}$ denoting that sources $h_1, \dots, h_n$, say in increasing order of number, are currently busy, a transmission by source $h \notin H$ is

$$\begin{cases} \text{accepted and initiated with probability: } \beta(h|H) \\ \text{rejected and lost with probability: } 1 - \beta(h|H) \end{cases}$$

Here, we make the natural assumption that these acceptance probabilities can only become smaller if more sources are busy, i.e. for all h, s and H :

$$\beta(h|H+s) \leq \beta(h|H) \quad (3.1)$$

When lost the source remains idle to schedule a new transmission request. By this latter blocking probability we can model various aspects such as:

(1) (Slotted ALOHA)

In slotted ALOHA, transmissions are time-slotted in time-slots of length Δ and take place along a single transmission switch. As this switch can handle only one task (acceptation or completion) at a time, a transmission can be accepted only if none of the ongoing transmissions is completed during that same time-slot. Hence, with $w_s = (1 - e^{-\Delta \gamma_s})$ the probability that source s completes its transmission during a time-slot of length Δ we have:

$$\beta(h|H) = \prod_{s \in H} (1 - w_s) = e^{-\Delta \sum_{s \in H} \gamma_s} \quad (3.2)$$

- (ii) Common memory utilization As another example, a busy sources may communicate with (retrieve data from or store data at) some memory device M, say during a fraction w_s of its busy time. However, to start a transmission a source must first retrieve some information from this memory. As this memory can communicate with only one source at a time, none of the busy sources may thus communicate with this memory upon transmission request. This is modeled by:

$$\beta(h|H) = \prod_{s \in H} (1-w_s). \quad (3.3)$$

Both examples can be shown (cf. [18], [19]) to be reversible (cf. [8] for a definition) if and only if the source characteristics w_s are the same for all sources, i.e. $w_s = w$ for some w and all s . In that case, the steady state distribution $\pi(H)$ is given by (cf. [18], [19]).

$$\pi(H) = c(1-w)^{[(M-[H])!-1]} \prod_{h \in H} [\gamma_h/\mu_h]. \quad (3.4)$$

where c is a normalizing constant and $[H]$ the cardinality of H . For the case with unequal values w_s , though, no simple explicit expression seems to be available. As numerical computation becomes computationally most expensive for reasonable large systems, approximations that can reduce the computation are thus of interest.

3.2 Approximation

Below we merely present the idea of a general approximation procedure which is developed in [19] and apply it to a system with four sources. The insight and underlying details are based on so-called Möbius expansions as investigated in more detail in this reference.

General idea

Consider a continuous-time Markov process with state description H and transition rates $q(H, H')$ such that $q(H, H') = 0$ if $||H'| - |H|| \geq 2$. That is, only one source can change its status at a time. When the process is reversible the stationary distribution $\pi(\cdot)$ can be expressed as (cf. [19]):

$$\pi(H) = \exp V(H)$$

with

$$V(H) = \sum_{B \in H, |B| \leq K} V(B) \sum_{j=0}^{K-|B|} (-1)^j \binom{|B|-|B|}{j} \quad (3.5)$$

where K is some specific integer that follows from the transition structure. (Roughly speaking, K is the value such that states which differ in more than K sources do not influence each other in terms of a potential interpretation similar to physical interactions). As a result, in the reversible case, the stationary probabilities of all states can be expressed in states with cardinality less or equal than K. Here one must typically think of K to be small, say K=1, 2 or 3. For example, with K=2 and M=4 we would obtain:

$$\begin{cases} \pi(i,j,k) = \pi(\emptyset)\pi(i,j)\pi(i,k)\pi(j,k)/[\pi(i)\pi(j)\pi(k)] \\ \pi(1,2,3,4) = [\pi(\emptyset)]^3 \frac{\pi(1,2)\pi(1,3)\pi(1,4)\pi(2,3)\pi(2,4)\pi(3,4)}{[\pi(1)\pi(2)\pi(3)\pi(4)]^2} \end{cases} \quad (3.6)$$

In the non-reversible case, however, these latter relations fail but still do seem reasonable as approximations. Roughly speaking: the approximation then comes down to ignoring interactions of states which differ in more than K sources.

Application Now reconsider the model of section 3.1 and for presentational convenience assume M=4. In the case of unequal values w_g a reasonable approximation $\tilde{\pi}(\cdot)$ thus seems to be suggested by the reduced set of global balance equations:

$$\begin{cases} \tilde{\pi}(H) \sum_{H'} q(H, H') = \sum_{H'} \tilde{\pi}(H') q(H', H), & \text{for } H \text{ with } |H| \leq 2 \\ (3.6) \text{ with } \pi(\cdot) \text{ replaced by } \tilde{\pi}(\cdot), & \text{for } H \text{ with } |H| > 2 \end{cases} \quad (3.7)$$

Here,

$$q(H, H') = \begin{cases} \gamma_h \beta(h|H) & H' = H \cup \{h\} \quad (h \notin H) \\ \gamma_h & H' = H \setminus \{h\} \quad (h \in H) \end{cases} \quad (3.8)$$

and $q(H, H')=0$ otherwise. Numerical computations have shown that this approximation is quite reasonable for the total distribution. However, no theoretical and guaranteed error bounds on the accuracy of such approximations have been reported. To this end, we aim to investigate the conditions of corollary 2.2.

3.3 Estimation of bias-terms:R

We need to verify the essential condition (2.19). Here we note that we can simply identify a state i with a state H so that all results of section 2 can be adopted directly for this multi-dimensional application. As performance measure of interest we aim to evaluate the throughput of the system by:

$$\langle \bar{r} \rangle (H) = \sum_{s \in H} \gamma_s \beta(s|H).$$

Further, we choose Q by

$$Q = \sum_h [\gamma_h + \mu_h]. \quad (3.9)$$

Let $V_t(\cdot)$ be defined by (2.5), (2.6) and (2.7) with the transition rates (3.8) substituted. As a result, we only need to verify (2.19) for states of the form: $i=H$ and $j=H \setminus \{h\}$ and $j=H \cup \{h\}$. To this end, for convenience write $H-h=H \setminus \{h\}$ and $H+h=H \cup \{h\}$.

Lemma 3.1 With $\mu_h \geq \sum_s \gamma_s$ for all $h \in H$, we have for all $H, H+h$ and $t \geq 0$:

$$0 \leq V_t(H) - V_t(H+h) \leq 1 \quad (3.10)$$

Proof This will be established by induction in t . As $V_t(\cdot) = 0$, (3.10) holds for $t=0$. Suppose that (3.10) holds for $t \leq m$ and $H, H+h$. The following relations then follow by expressing $V_{m+1}(H+h)$ and $V_{m+1}(H)$ as according to (2.12). Herein, we note in advance that some terms are rewritten or artificially added and subtracted (e.g., the term $\gamma_h Q^{-1} \beta(h|H) V_m(H+h)$ in the first relation) or split (e.g. $\beta(s|H) = \beta(s|H+h) + [\beta(s|H) - \beta(s|H+h)]$ also in the first relation), in order to obtain terms with equal coefficients that can be compared pairwise later on. Further, we recall (3.1).

Then by virtue of (2.12):

$$V_{m+1}(H)$$

-

$$\sum_{s \in H} \gamma_s Q^{-1} \beta(s|H)$$

$$\sum_{s \in H} \mu_s Q^{-1} V_m(H-s) +$$

$$\sum_{s \in H} \gamma_s Q^{-1} \beta(s|H) V_m(H+s) +$$

$$\left[1 - \sum_{s \in H} \mu_s Q^{-1} - \sum_{s \in H} \gamma_s Q^{-1} \right] V_m(H)$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H) + \gamma_h Q^{-1} \beta(h|H)$$

$$\sum_{s \in H} \mu_s Q^{-1} V_m(H-s) + \mu_h Q^{-1} V_m(H) +$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H+h) V_m(H+s) +$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \left[\beta(s|H) - \beta(s|H+h) \right] V_m(H+s) + \gamma_h Q^{-1} \beta(h|H) V_m(H+h) +$$

$$\left[1 - \sum_{s \in H+h} \mu_s Q^{-1} - \sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H) - \gamma_h Q^{-1} \beta(h|H) \right] V_m(H).$$

And similarly

$$V_{m+1}(H+h)$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H+h) +$$

$$\sum_{s \in H} \mu_s Q^{-1} V_m(H+h-s) + \mu_h Q^{-1} V_m(H)$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H+h) V_m(H+h+s) +$$

$$\left[1 - \sum_{s \in H+h} \mu_s Q^{-1} - \sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H+h) \right] V_m(H+h)$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H+h)$$

$$\mu_h Q^{-1} V_m(H) + \gamma_h Q^{-1} \beta(h|H) V_m(H+h) +$$

$$\sum_{s \in H} \mu_s Q^{-1} V_m(H+h-s) +$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H+h) V_m(H+h+s) +$$

$$\sum_{s \in H+h} \gamma_s Q^{-1} \left[\beta(s|H) - \beta(s|H+h) \right] V_m(H+h)$$

$$\left[1 - \sum_{s \in H+h} \mu_s Q^{-1} - \sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H) - \gamma_h Q^{-1} \beta(h|H) \right] V_m(H+h).$$

Hence

$$\begin{aligned}
 & V_{m+1}(H) - V_{m+1}(H+h) \\
 & = \\
 & \gamma_h Q^{-1} \beta(h|H) + \\
 & \sum_{s \in H+h} \gamma_s Q^{-1} [\beta(s|H) - \beta(s|H+h)] \\
 & \gamma_h Q^{-1} \beta(h|H) [V_m(H+h) - V_m(H+h)] + \\
 & \mu_h Q^{-1} [V_m(H) - V_m(H)] + \\
 & \sum_{s \in H} \mu_s Q^{-1} [V_m(H-s) - V_m(H+h-s)] \\
 & \sum_{s \in H+h} \gamma_s Q^{-1} \beta(s|H+h) [V_m(H+s) - V_m(H+h+s)] + \\
 & \sum_{s \in H+h} \gamma_s Q^{-1} [\beta(s|H) - \beta(s|H+h)] [V_m(H+s) - V_m(H+h)] + \\
 & [1 - \sum_{s \in H+h} \mu_h Q^{-1} + \sum_{s \in H} \gamma_s Q^{-1} \beta(s|H)] [V_m(H) - V_m(H+h)] \tag{3.11}
 \end{aligned}$$

where indeed the third and fourth term are equal to 0 but kept in for clarity and the use of an argument below. First note that we can write

$$V_m(H+s) - V_m(H+h) = [V_m(H+s) - V_m(H)] + [V_m(H) - V_m(H+h)]. \tag{3.12}$$

The first term of the right hand side of (3.12) is nonpositive but estimated from below by -1 as per induction assumption (3.10) for $t=m$. This is compensated by the term:

$$\sum_{s \in H+h} \gamma_s Q^{-1} [\beta(s|H) - \beta(s|H+h)]$$

which equals the coefficient of (3.12) in (3.11). By substituting the lower estimate 0 from (3.10) for $t=m$ in all other terms, the right hand side of (3.12) is thus estimated from below by 0. That is, we have shown the lower estimate 0 of (3.10) for $t=m+1$.

To conclude the upper estimate 1, now recall that the third and fourth term are equal to 0 while also $\mu_h \geq \sum_s \gamma_s$. As a consequence, these 0-terms can compensate for the first two additional positive terms. More precisely, by

estimating the right hand side of (3.12) from above by $V_m(H) - V_m(H+h)$, as justified by the hypothesis (3.10) for $t=m$, substituting the upper estimate 1 from (3.10) in all terms and noting that all coefficients sum up to 1, the right hand side of (3.11) is estimated from above by 1. That is, we have also shown the upper estimate 1 of (3.10) for $t=m+1$.

Induction completes the proof. □

3.3 Numerical examples

We can now apply corollary 2.2 with $R=1$. The value Δ is thus to be computed by substituting the approximations as per (3.7). Numerical illustration is provided below. Here, we note that realistically for the applications as described in section 3.1, most notably the slotted-ALOHA model, the w -values should be thought of as being small, in which case the results are quite reasonable (in the order of 1%). But also larger less realistic w -values are included for the purpose of testing. Roughly, the results indicate that the error bound as based on (2.17) will be quite rough compared to the exact error bound but yet quite reasonable, in the order of a few percent, as a 100% secure bound on the imprecision involved. The exact performance value g is included for comparison and is obtained by numerically solving the system.

Numerical examples

$$(\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1)$$

$$(\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1)$$

g : expected number of idle sources

$\bar{g}$: approximate value based on $\bar{\pi}$ as per (3.7)

Δ : error bound for $|g - \bar{g}|$ as based on (2.17) and (2.19) ($R=1$)

Example 1

$w_1 = .5$	$g = 2.628953$	$\Delta = .114 (4.3\%)$
$w_2 = .5$	$\bar{g} = 2.628985$	
$w_3 = .5$		
$w_4 = .45$		

Example 2

$w_1 = .2$	$g = 2.299938$	$\Delta = .063 \text{ (2.7\%)}$
$w_2 = .2$	$\bar{g} = 2.299955$	
$w_3 = .2$		
$w_4 = .25$		

Example 3

$w_1 = .2$	$g = 2.316404$	$\Delta = .083 \text{ (3.6\%)}$
$w_2 = .25$	$\bar{g} = 2.316425$	
$w_3 = .2$		
$w_4 = .25$		

Example 4

$w_1 = .1$	$g = 2.252328$	$\Delta = .149 \text{ (6.6\%)}$
$w_2 = .15$	$\bar{g} = 2.252416$	
$w_3 = .2$		
$w_4 = .25$		

Example 5

$w_1 = .14$	$g = 2.846528$	$\Delta = .071 \text{ (2.5\%)}$
$w_2 = .16$	$\bar{g} = 2.846578$	
$w_3 = .18$		
$w_4 = .20$		

Example 6

$w_1 = .04$	$g = 2.103492$	$\Delta = .047 \text{ (2.3\%)}$
$w_2 = .06$	$\bar{g} = 2.103497$	
$w_3 = .08$		
$w_4 = .10$		

Example 7

$w_1 = .01$	$g = 2.037369$	$\Delta = .021 (1.0\%)$
$w_2 = .02$	$g = 2.037371$	
$w_3 = .03$		
$w_4 = .04$		

Example 8

$w_1 = .01$	$g = 2.0224361$	$\Delta = .010 (.5\%)$
$w_2 = .02$	$g = 2.0224364$	
$w_3 = .01$		
$w_4 = .02$		

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