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Processes**

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INSENSITIVITY IN DISCRETE TIME GENERALIZED SEMI-MARKOV PROCESSES

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Abstract

In this paper we provide a criterion for a discrete time Generalized Semi-Markov Process to be insensitive.

Keywords: Insensitivity, Generalized Semi-Markov Process, Product Form

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1. Introduction

Many authors have discussed criteria for insensitivity in continuous time Generalized Semi-Markov Processes (GSMPs) or other similar structures (see, for example, König and Jansen [6], Schassberger [8], [9], [10] and [11], Kelly [7], Franken et al. [4], Burman [1], Henderson [5], Whittle [13] and [14], and Taylor [11]). Such criteria usually take the form of partial balance equations which must be satisfied by the GSMP when all its lifetimes are taken to be exponential.

However the subject of insensitivity in discrete time GSMPs appears to have been discussed by only a few authors. Daduna and Schassberger [2] showed that a round robin queue with general service times has a product form stationary distribution. In [3] they also discussed product form in a discrete time network of queues with a first come first served discipline at nodes with geometric service times and so-called doubly stochastic disciplines at nodes with type dependent deterministic service times. In both cases their queues were assumed to operate according to quite restrictive rules, principally disallowing the possibility of multiple lifetimes dying at the same time.

In this paper we use a general model for a discrete time GSMP and derive necessary and sufficient criteria for there to exist an invariant measure with product form. Unlike the queueing network model of Daduna and Schassberger it is possible for multiple lifetimes to die simultaneously in our GSMP. The model is set up in Section 2 and the main results proved in Sections 3 and 4. Some examples are discussed in Section 5.

2. The model

The model used in this section is a discrete time version of that discussed by König and Jansen [6] and Schassberger [10].

Consider a discrete time process which moves on a set of states G . Incorporated in each state $g \in G$ are active elements from a set S . With a customary abuse of notation we use g to denote also the set of active elements associated with the state g . When the element σ becomes alive, it is allocated a nominal lifetime sampled from a discrete distribution $F_\sigma(\cdot)$ with probability mass function $f_\sigma(\cdot)$ and with finite mean μ_σ independently of any other lifetimes which may become alive at that time. At each time point, with probability $c(s, g)$ each element of a subset $s \subseteq g$ has one unit worked off its residual lifetime and the lifetimes of the elements of $g \setminus s$ are unaffected. If the residual service requirement of one or more elements is reduced to zero at this time, then those elements will die and the process will make a transition to a state h . If the active elements of the subset $s_2 \subseteq s$ die, the process moves to the state $h \supseteq g \setminus s_2$ with probability $p(g, s_2, s, h)$ with the set $s_1 = h \setminus (g \setminus s_2)$ of elements being activated. We write $s_0 = s \setminus s_2$ for the set of elements which have a unit worked off their residual lifetimes without these thereby being reduced to zero.

Remark

The possibilities $g = h$ and $s = \emptyset$ are not excluded in the transitions described above. In particular allowing the choice $s = \emptyset$ allows greater economy in modelling than is offered by the standard continuous time GSMP model (see Schassberger [10]). In the standard model the set S of lifetimes is divided up into sets S' of exponentially distributed lifetimes and sets S^* of generally distributed lifetimes. In many situations the exponentially distributed lifetimes are used as "triggers" to activate generally distributed lifetimes. (For example, a Poisson arrival to a queue activates a generally distributed service time.) The necessity to do this arises from the fact that in the standard continuous time formulation transitions only occur when a lifetime dies. In contrast our discrete time formulation allows lifetimes to be created at a time point even if no other lifetimes have died. Such a formulation is equivalent to having a set of geometrically distributed lifetimes triggering

these transitions but is more efficient.

Supplement the states g with a vector of lifetimes x , where $x_\sigma \geq 1$ is the residual lifetime of an active element $\sigma \in g$ and $x_\sigma = 0$ for $\sigma \in S \setminus g$. Then the process with supplemented states is a Markov chain. A transition from state (g, x) to state (h, y) is possible only if g is of the form $(h \setminus s_1) \cup s_2$ with the set of residual lifetimes x of the form

$$x = y + e(s_0) - y_1 + e(s_2)$$

for some $s_0 \subseteq g \setminus s_1$. (Here $e(t)$ is the vector with a one in each position $\sigma \in t$ and zeros elsewhere and y_1 is the vector of freshly created lifetimes corresponding to s_1 .)

In this notation, the Markov chain has transition probabilities

$$q((g, x), (h, y)) = c(s_2 \cup s_0, (h \setminus s_1) \cup s_2) p((h \setminus s_1) \cup s_2, s_2, s_2 \cup s_0, h) \prod_{\sigma \in s_1} f_\sigma(y_\sigma) \quad (2.1)$$

and equilibrium equations

$$\begin{aligned} \pi(h, y) = & \sum_{s_1 \subseteq h} \sum_{s_0 \subseteq h \setminus s_1} \sum_{s_2 \subseteq S \setminus (h \setminus s_1)} \pi((h \setminus s_1) \cup s_2, x) \\ & \times c(s_0 \cup s_2, (h \setminus s_1) \cup s_2) p((h \setminus s_1) \cup s_2, s_2, s_0 \cup s_2, h) \prod_{\sigma \in s_1} f_\sigma(y_\sigma) \end{aligned} \quad (2.2)$$

where $x = y + e(s_0) - y_1 + e(s_2)$.

We are interested in the case when there exists a solution to equations (2.2) with product form over the supplementary variables. That is to say, when

$$\pi(h, y) = \pi(h) \left[\prod_{\sigma \in h} \frac{r_\sigma(y_\sigma)}{\mu_\sigma} \right] \quad (h \in G), \quad (2.3)$$

where $r_\sigma(y) = \sum_{n \geq y} f_\sigma(n)$. Before proceeding to examine this case, we shall derive Proposition 1 below which conveniently encapsulates the notion of insensitivity. To this end, we first define the set of lifetimes

$$T = \{\sigma \in h : \mu_\sigma = 1\},$$

and, for each $\sigma \in S$, the set of distributions

$$\Phi_\sigma = \{f_\sigma(\cdot) : \sum_{l=1}^{\infty} l f_\sigma(l) = \mu_\sigma\}.$$

Proposition 1

Let (h, y) be a supplemented state for which $y_\sigma \geq 1 \forall \sigma \in h \setminus T$ and $y_\sigma = 1 \forall \sigma \in h \cap T$. Then it is necessary and sufficient for $(a_k; k \subseteq h)$ to be a solution to

$$\sum_{k \subseteq h} a_k \left[\prod_{\sigma \in k} f_\sigma(y_\sigma) \right] \prod_{\sigma \in h \setminus k} r_\sigma(y_\sigma + 1) = 0 \quad \forall f_\sigma \in \Phi_\sigma (\sigma \in h) \quad \forall k \subseteq h, \quad (2.4)$$

with (a_k) not depending on $\{f_\sigma\}$ except possibly through the means $\{\mu_\sigma; \sigma \in h\}$, that

$$a_k = 0 \quad \forall T \subseteq k \subseteq h. \quad (2.5)$$

We observe that the constraint that f_σ is a probability mass function with mean μ_σ can be expressed as the conditions that r_σ is a non-negative and monotone non-increasing function satisfying

$$r_\sigma(1) = 1 \quad (2.6)$$

and

$$\sum_{l \geq 1} r_\sigma(l) = \mu_\sigma. \quad (2.7)$$

If $\mu_\sigma = 1$ this necessarily means that

$$r_\sigma(l) = 0 \quad \forall l > 1. \quad (2.8)$$

3. Proof of Proposition 1

Firstly note that $\prod_{\sigma \in h \setminus k} r_\sigma(y_\sigma + 1) = 0$ for all $k \not\supseteq T$ on account of (2.8). Sufficiency follows immediately.

For necessity observe that equations (2.5) may be written as

$$\sum_{T \subseteq k \subseteq h} a_k \left[\prod_{\sigma \in k} \{r_\sigma(y_\sigma) - r_\sigma(y_\sigma + 1)\} \right] \prod_{\sigma \in h \setminus k} r_\sigma(y_\sigma + 1) = 0 \quad (k \subseteq h). \quad (3.1)$$

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Since $\mu_\sigma > 1$ for all $\sigma \in h \setminus T$ there exists for each such σ a net $(f_{\sigma,t}; t \in I) \subseteq \Phi_\sigma$ such that

(a) for each t

$$r_{\sigma,t}(l-1) > r_{\sigma,t}(l) > r_{\sigma,t}(l+1) > r_{\sigma,t}(l+2)$$

for $l = y_\sigma$ and also for some $l = y'_\sigma > y_\sigma + 1$;

(b) by varying the choice of t , the values $r_{\sigma,t}(l)$ and $r_{\sigma,t}(l+1)$ can be varied continuously and independently over non-trivial intervals J_σ , J_σ^* , with the mean constraint (2.7) being maintained through simultaneous variations of $r_{\sigma,t}(y'_\sigma)$, $r_{\sigma,t}(y'_\sigma + 1)$.

It follows from the elementary properties of analytic functions in several complex variables that the coefficient of each distinct multinomial

$$\left(\prod_{\sigma \in m} z_\sigma \right) \prod_{\sigma \in h \setminus m} z'_\sigma$$

in the multilinear form

$$\sum_{T \subseteq k \subseteq h} a_k \left[\prod_{\sigma \in k} (z_\sigma - z'_\sigma) \right] \prod_{\sigma \in h \setminus k} z'_\sigma$$

must vanish, that is,

$$\sum_{m: k \subseteq m \subseteq h} a_m (-1)^{|m \setminus k|} = 0 \quad (\forall T \subseteq k \subseteq h). \quad (3.2)$$

An induction on $|h \setminus k|$ establishes that each $a_k = 0$. The basis, with $k = h$, is trivial. For the inductive step, the assumption $a_m = 0$ for all $m \supset k$ gives $a_k = 0$ immediately from (3.2). $\square$

The proposition can be paraphrased as saying that the functionals

$$\left[\prod_{\sigma \in k} f_\sigma(y_\sigma) \right] \prod_{\sigma \in h \setminus k} r_\sigma(y_\sigma + 1)$$

for $T \subseteq k \subseteq h$ are linearly independent.

4. The balance conditions

Theorem 1

A necessary and sufficient condition for the supplemented discrete-time GSMP to have an invariant measure of the form (2.3) is that the equations

$$\frac{\pi(\mathbf{h})}{\prod_{s \in \mathbf{h}} \mu_s} = \sum_{s_1 \subseteq \mathbf{k}} \sum_{s_2 \subseteq S \setminus (\mathbf{h} \setminus s_1)} \frac{\pi((\mathbf{h} \setminus s_1) \cup s_2)}{\prod_{s \in (\mathbf{h} \setminus s_1) \cup s_2} \mu_s} \Psi(\mathbf{h} \setminus \mathbf{k}, s_2, (\mathbf{h} \setminus s_1) \cup s_2) \quad (4.1)$$

have a solution when taken over all $\mathbf{h}$ and $T \subseteq \mathbf{k} \subseteq \mathbf{h}$, where

$$\Psi(\mathbf{h} \setminus \mathbf{k}, s_2, (\mathbf{h} \setminus s_1) \cup s_2) = \sum_{s_0 \subseteq \mathbf{h} \setminus \mathbf{k}} c(s_0 \cup s_2, (\mathbf{h} \setminus s_1) \cup s_2) p(\mathbf{h} \setminus s_1 \cup s_2, s_2, s_0 \cup s_2, \mathbf{h}).$$

Proof

States $(\mathbf{h}, \mathbf{y})$ with $y_\sigma > 1$ for $\sigma \in \mathbf{h} \cap T$ are transient in the supplemented GSMP. Thus we need only consider states in which $y_\sigma = 1$ for $\sigma \in \mathbf{h} \cap T$. On substitution of the product form (2.3) into the equilibrium equations (2.2) for such states, noting that $\prod_{\sigma \in s_2} r_\sigma(1) = 1$, the right hand side becomes

$$\begin{aligned} & \sum_{s_1 \subseteq \mathbf{h}} \sum_{s_0 \subseteq \mathbf{h} \setminus s_1} \sum_{s_2 \subseteq S \setminus (\mathbf{h} \setminus s_1)} \frac{\pi((\mathbf{h} \setminus s_1) \cup s_2)}{\prod_{s \in (\mathbf{h} \setminus s_1) \cup s_2} \mu_s} c(s_0 \cup s_2, (\mathbf{h} \setminus s_1) \cup s_2) \\ & \times p((\mathbf{h} \setminus s_1) \cup s_2, s_2, s_0 \cup s_2, \mathbf{h}) \left[\prod_{\sigma \in s_1} f_\sigma(y_\sigma) \right] \left[\prod_{\sigma \in s_0} r_\sigma(y_\sigma + 1) \right] \prod_{\sigma \in \mathbf{h} \setminus (s_1 \cup s_0)} r_\sigma(y_\sigma). \end{aligned} \quad (4.2)$$

Now write

$$\begin{aligned} \prod_{\sigma \in \mathbf{h} \setminus (s_1 \cup s_0)} r_\sigma(y_\sigma) &= \prod_{\sigma \in \mathbf{h} \setminus (s_1 \cup s_0)} [r_\sigma(y_\sigma + 1) + f_\sigma(y_\sigma)] \\ &= \sum_{w \subseteq \mathbf{h} \setminus (s_1 \cup s_0)} \left[\prod_{\sigma \in w} f_\sigma(y_\sigma) \right] \prod_{\sigma \in \mathbf{h} \setminus (s_1 \cup s_0 \cup w)} r_\sigma(y_\sigma + 1) \end{aligned}$$

in (4.2) and change the order of summation to get

$$\begin{aligned} & \sum_{w \subseteq \mathbf{h}} \sum_{s_1 \subseteq \mathbf{h} \setminus w} \sum_{s_0 \subseteq \mathbf{h} \setminus (w \cup s_1)} \sum_{s_2 \subseteq S \setminus (\mathbf{h} \setminus s_1)} \frac{\pi((\mathbf{h} \setminus s_1) \cup s_2)}{\prod_{s \in (\mathbf{h} \setminus s_1) \cup s_2} \mu_s} c(s_0 \cup s_2, (\mathbf{h} \setminus s_1) \cup s_2) \\ & \times p((\mathbf{h} \setminus s_1) \cup s_2, s_2, s_0 \cup s_2, \mathbf{h}) \left[\prod_{\sigma \in s_1 \cup w} f_\sigma(y_\sigma) \right] \prod_{\sigma \in \mathbf{h} \setminus (s_1 \cup w)} r_\sigma(y_\sigma + 1). \end{aligned}$$

Make the substitution $k = s_1 \cup w$ in (4.3) to derive

$$\sum_{k \subseteq h} \sum_{s_1 \subseteq k} \sum_{s_0 \subseteq h \setminus k} \sum_{s_2 \subseteq S \setminus (h \setminus s_1)} \frac{\pi((h \setminus s_1) \cup s_2)}{\prod_{s \in (h \setminus s_1) \cup s_2} \mu_s} c(s_0 \cup s_2, (h \setminus s_1) \cup s_2) \quad (4.3)$$

$$\times p((h \setminus s_1) \cup s_2, s_2, s_0 \cup s_2, h) \left[\prod_{\sigma \in k} f_\sigma(y_\sigma) \right] \prod_{\sigma \in h \setminus k} r_\sigma(y_\sigma + 1).$$

Also expand the left hand side of (2.2) with (2.3) substituted to get

$$\frac{\pi(h)}{\prod_{s \in h} \mu_s} \sum_{k \subseteq h} \left[\prod_{\sigma \in k} f_\sigma(y_\sigma) \right] \prod_{\sigma \in h \setminus k} r_\sigma(y_\sigma + 1). \quad (4.4)$$

By Proposition 1 a necessary and sufficient condition for (4.3) to equal (4.4) for all possible vectors of distributions with $f_\sigma(\cdot)$ selected from Φ_σ , is that the corresponding coefficients of $\left[\prod_{\sigma \in k} f_\sigma(y_\sigma) \right] \prod_{\sigma \in h \setminus k} r_\sigma(y_\sigma + 1)$ are equal for $k \supseteq T$. The balance equations (4.1) follow. $\square$

Corollary 1

Let $\pi(h)$ be a solution to equations (4.1) and let $\pi(h, y)$ be the corresponding supplemented solution when the σ component of the vector of distributions is selected from Φ_σ . Then

$$\sum_y \pi(h, y) = \pi(h). \quad (4.5)$$

Proof

The result follows easily by using Theorem 1 and summing equation (2.3). $\square$

Corollary 1 gives us a criterion for insensitivity of the discrete time GSMP. If a solution can be found to equations (4.1) which is a probability distribution then the marginal distribution that the process is in state h remains invariant providing each distribution F_σ is selected from Φ_σ . In many natural special cases balance equations (4.1) reduce to simpler equations. In the next section we discuss some of these.

5. Some special cases

5.1 The case when all lifetimes are served

A discrete time GSMP which operates so that every lifetime that is alive receives a unit of service at each time point can be modelled using the structure of section 2 by putting

$$c(s, h) = \delta_{sh}$$

where

$$\delta_{sh} = \begin{cases} 1 & \text{if } s = h \\ 0 & \text{if } s \neq h. \end{cases}$$

It is also redundant to carry the argument s in the routing probabilities, thus we write $p((h \setminus s_1) \cup s_2, s_2, h)$ for $p((h \setminus s_1) \cup s_2, s_2, h, h)$. It now follows that

$$\Psi(h \setminus k, s_2, (h \setminus s_1) \cup s_2) = \delta_{ks_1} p((h \setminus s_1) \cup s_2, s_2, h)$$

and so (4.1) reduces to

$$\frac{\pi(h)}{\prod_{s \in h} \mu_s} = \sum_{s_2 \subseteq S \setminus (h \setminus k)} \frac{\pi((h \setminus k) \cup s_2)}{\prod_{s \in (h \setminus k) \cup s_2} \mu_s} p((h \setminus k) \cup s_2, s_2, h) \quad (5.1)$$

which must be satisfied for all $k \supseteq T$. Note that $\prod_{s \in h \setminus k} \mu_s^{-1}$ appears on both sides of equation (5.1). If these terms were cancelled out the equations would be

$$\frac{\pi(h)}{\prod_{s \in k} \mu_s} = \sum_{s_2 \subseteq S \setminus (h \setminus k)} \frac{\pi((h \setminus k) \cup s_2)}{\prod_{s \in s_2} \mu_s} p((h \setminus k) \cup s_2, s_2, h). \quad (5.2)$$

Readers familiar with insensitivity results in continuous time GSMPs may see some similarity between equation (5.2) and the partial balance equations, necessary and sufficient for product form in standard continuous time GSMPs. The left hand side is analogous to the flux out of state h due to the death of the lifetimes in k and the right hand side is analogous to the flux into state h due to the birth of the lifetimes in k .

5.2 The case when only one lifetime is served at a time

Many models of practical interest (for example, the discrete time version of the processor sharing queue) have a single server who, at each time point, allocates his service to one of the existing lifetimes according to a probability distribution. Such GSMPs can be modelled in the current framework by putting $c(s, h) = 0$ if $|s| > 1$. With this restriction

$$\Psi(h \setminus k, s_2, (h \setminus s_1) \cup s_2) = \begin{cases} c(s_2, (h \setminus s_1) \cup s_2) p((h \setminus s_1) \cup s_2, s_2, s_2, h) & \text{if } |s_2| = 1 \\ \sum_{s_0 \in h \setminus k} c(s_0, h \setminus s_1) p((h \setminus s_1), \emptyset, \{s_0\}, h) & \text{if } s_2 = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

For notational convenience below we write s for

Equation (4.1) becomes

$$\begin{aligned} \frac{\pi(h)}{\prod_{s \in h} \mu_s} &= \sum_{s_1 \subseteq k} \frac{\pi(h \setminus s_1)}{\prod_{s \in h \setminus s_1} \mu_s} \sum_{s_0 \in h \setminus k} c(s_0, h \setminus s_1) p((h \setminus s_1), \emptyset, s_0, h) \\ &+ \sum_{s_1 \subseteq k} \sum_{s_2 \in S \setminus (g \setminus s_1)} \frac{\pi((h \setminus s_1) \cup s_2)}{\prod_{s \in (h \setminus s_1) \cup s_2} \mu_s} c(s_2, (h \setminus s_1) \cup s_2) p((h \setminus s_1) \cup s_2, s_2, s_2, h). \end{aligned} \quad (5.3)$$

5.3 The models of Daduna and Schassberger

In [2] and [3] Daduna and Schassberger consider special cases of GSMPs in which it is not permitted for a transition to occur when the lifetime of more than one element of h is reduced to zero at the same time point. The mechanism employed by Daduna and Schassberger is to assume that such elements are allocated a new lifetime sampled from the original distribution. In our current model this is equivalent to putting

$$p(h, s_2, s_0 \cup s_2, h') = \delta_{hh'}$$

for all s_0 and s_2 with $|s_2| > 1$. Substituting this into equation (4.1) yields

$$\begin{aligned} \frac{\pi(h)}{\prod_{s \in h} \mu_s} &= \sum_{s_1 \subseteq k} \sum_{s_2 \subseteq S \setminus (h \setminus s_1), |s_2| \leq 1} \frac{\pi((h \setminus s_1) \cup s_2)}{\prod_{s \in (h \setminus s_1) \cup s_2} \mu_s} \Psi(h \setminus k, s_2, (h \setminus s_1) \cup s_2) \\ &+ \sum_{s_1 \subseteq k, |s_2| > 1} \frac{\pi(h)}{\prod_{s \in h} \mu_s} \sum_{s_0 \subseteq h \setminus k} c(s_0 \cup s_1, h). \end{aligned} \quad (5.4)$$

Note that, for all h and $k \subseteq h$,

$$\sum_{s_1 \subseteq k} \sum_{s_0 \subseteq h \setminus k} c(s_0 \cup s_1, h) = 1.$$

Using this, equation (5.4) can be rearranged to get

$$\begin{aligned} \frac{\pi(h)}{\prod_{s \in h} \mu_s} \sum_{s_1 \subseteq k, |s_2| \leq 1} \sum_{s_0 \subseteq h \setminus k} c(s_0 \cup s_1, h) \\ = \sum_{s_1 \subseteq k} \sum_{s_2 \subseteq S \setminus (h \setminus s_1), |s_2| \leq 1} \frac{\pi((h \setminus s_1) \cup s_2)}{\prod_{s \in (h \setminus s_1) \cup s_2} \mu_s} \Psi(h \setminus k, s_2, (h \setminus s_1) \cup s_2). \end{aligned} \quad (5.5)$$

If we now assume, as in example 5.1, that $c(s, h) = \delta_{sh}$ then the left hand side of equation (5.5) is zero whenever $|k| > 1$. Employing the notation of Example 5.1, the right hand side becomes

$$\sum_{s_2 \subseteq S \setminus (h \setminus k), |s_2| \leq 1} \frac{\pi((h \setminus k) \cup s_2)}{\prod_{s \in (h \setminus k) \cup s_2} \mu_s} p((h \setminus k) \cup s_2, s_2, h)$$

and it follows that the only way equation (5.5) can be satisfied is if $p((h \setminus k) \cup s_2, s_2, h) = 0$ whenever $|k| > 1$. Hence in these circumstances the GSMP can have product form only if, in addition to no transitions occurring when more than one lifetime dies, it is impossible for more than one lifetime to be created at the same time point.

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