

SERIE RESEARCH MEMORANDA

APPROXIMATE UNIFORMIZATION FOR
CONTINUOUS-TIME MARKOV CHAINS
WITH AN APPLICATION TO PERFORMABILITY ANALYSIS

Nico M. van Dijk

Research Memorandum 1988-54 Dec.'88



VRIJE UNIVERSITEIT
FACULTEIT DER ECONOMISCHE WETENSCHAPPEN
EN ECONOMETRIE
AMSTERDAM

APPROXIMATE UNIFORMIZATION FOR
CONTINUOUS-TIME MARKOV CHAINS
WITH AN APPLICATION TO PERFORMABILITY ANALYSIS

Nico M. van Dijk

Free University, Amsterdam, The Netherlands

Abstract An approximate version of the standard uniformization technique is introduced for application to continuous-time Markov chains with unbounded jump rates. This technique is shown to be asymptotically exact and an error bound for the order of its accuracy is provided. An illustrative queueing application is included.

Keywords Continuous-time Markov chains, jump rates, approximate uniformization, error bound



1 Introduction

Continuous-time Markov chains are widely applicable for modeling practical situations that evolve continuously in time with jumps or changes at specific epochs. Particularly, applications are found in communication (queueing theory, broadcasting), computer performance evaluation (computer networks), manufacturing (material handling systems, assembly lines), inventory theory, maintenance analysis and reliability.

In dealing with continuous time Markov chains the technique of uniformization, motivated by results in [2], has been shown to be a powerful tool for both computational and theoretical purposes over the last decades. Among various others, see for instance [13], p.110 for a short description of this technique. For example, uniformization has been fruitfully exploited for Markov decision processes both directly (cf. [8]) and indirectly such as by speeding up policy improvement schemes by artificially introducing extra decision epochs (cf. [14]).

The uniformization technique, however, is essentially based upon the assumption of uniformly bounded transition or jump rates. Unfortunately, this assumption is too restrictive in many applications with unbounded state spaces. For example, in an infinite queueing model the service or departure rate can be linear in the number of customers present, in an inventory system the demand rate may have unbounded peaks, and in reliability the death rate may be linear in the elapsed life time.

Recently, cf. [17], error bounds and rates of convergence have been provided for truncations of discrete time infinite Markov chains. These results could be applied to infinite queueing systems with bounded jump rates by applying the uniformization technique. Other truncation results for discrete time Markov chains and thus also continuous time Markov chains with bounded jump rates are given in [6], [7] and [20]. For infinite queueing systems with unbounded jump rates, however, the uniformization technique and thus also these discrete time results cannot be applied.

This paper, therefore, is concerned with an approximate uniformization technique that applies to continuous-time Markov chains also with unbounded jump rates. This technique is a truncated version of standard uniformization. As the state space is hereby truncated this technique is computationally applicable in a twofold manner. As a truncation of the transition rates and as a truncation, but not necessarily associated to the rate truncation, of the state space. An error bound for the accuracy will be established which is reciprocal in the truncation limit with a conditioning constant depending upon estimates for the bias terms of the underlying reward structure associated with the performance measure of interest.

To illustrate the approximate uniformization technique as well as how estimates for these bias terms can be obtained the results are applied to an infinite queueing system with breakdowns and unbounded service rates. An explicit error bound expression for this application will be provided.

First, as unbounded jump rates are involved, conditions are to be imposed so as to exclude explosions. To this end, results from [15] will be adopted (section 2). Next, the approximate technique and the key theorem are presented (section 3). The application to the breakdown system will then be examined in detail (section 4). An evaluation concludes the paper.

2 Model and preliminary results

The system under investigation is a continuous time Markov chain or Markov jump model with state space N and transition or jump rates $q(i,j)$ for a transition from state i into state j . Further, a reward rate $r(i)$ is incurred per unit of time when the system is in state i . First, let us introduce some notation and present assumptions.

$$q(i) = \sum_j q(i,j)$$

$$p(i,j) = q(i,j)/q(i)$$

Further, for $\mu: N \rightarrow \mathbb{R}$ with $\mu(\cdot) \geq 1$ and μ -non-decreasing, which will be called a bounding function hereafter, let

$$B^\mu = \{f: \mathbb{N} \rightarrow \mathbb{R} \mid \|f\|_\mu < \infty\}$$

where

$$\|f\|_\mu = \sup_i |f(i)/\mu(i)| \quad (f \in B^\mu).$$

Then B^μ is a Banach space endowed with norm $\|\cdot\|_\mu$. The following assumptions can then be stated.

Assumptions 2.1 For bounding functions μ_q and μ_r and constants K_q, K_r ,

$$\|q\|_{\mu_r} \leq K_q \quad (2.1)$$

$$\|r\|_{\mu_r} \leq K_r \quad (2.2)$$

and with

$$\mu_k = \mu_r (\mu_q)^k \quad (k=0,1,2)$$

and constants K, M , we have

$$\|\sum_j p(\cdot, j) \mu_k(j)\|_{\mu_k} \leq K \quad (k=0,1) \quad (2.3)$$

and either

$$\|\sum_{j>i} q(\cdot, j) [\mu_k(j) - \mu_k(\cdot)]\|_{\mu_k} \leq M \quad (k=0,1,2) \quad (2.4)$$

or

$$\|\sum_j q(\cdot, j) [\mu_k(j) - \mu_k(\cdot)]\|_{\mu_k} \leq M \quad (k=0,1,2) \quad (2.5)$$

The assumptions (2.1) and (2.2) allow for instance polynomially bounded jump and reward rates such as with

$$\mu_q(i) = (1+i)^m$$

$$\mu_r(i) = (1+i)^p$$

for certain m and p . The restrictive assumptions (2.3) and (2.4) or (2.5)

are purely technical and argued in detail in [15]. Roughly speaking they are needed to exclude explosions. As per this reference, however, various practical situations with unbounded jumprates are covered such as most notably when for some constants $\tau_q > 0$, $K_q < \infty$ and $\ell \in \mathbb{N}$:

$$q(i) \geq \tau_q \quad (\forall i)$$

$$q_+(i) = \sum_{j>i} q(i,j) \leq K_q \quad (\forall i) \quad (2.6)$$

$$q(i, i+k) = 0 \quad \text{for all } k > \ell \quad (\forall i)$$

which are satisfied in many one-dimensional queueing applications such as an $M|M|\infty$ -queue. Under the above assumptions 2.1 the following lemma can now be concluded. Roughly, it states that marginal as well as total expected reward functions over a finite time length are properly defined by a deterministic integral equation, completely similar to the standard bounded jump and reward case as in [1] or [5].

Lemma 2.1 For any μ_k -bounded function f , $k=0,1$, the function $A f$ is properly defined by:

$$A f(i) = \sum_j q(i,j) [f(j) - f(i)] \quad (2.7)$$

and

$$\|A f\|_{\mu_{k+1}} \leq (K+1)K_q \|f\|_{\mu_k} \quad (2.8)$$

Further, for any finite epoch S and any μ_r -bounded function r there exists a unique μ_r -bounded family $(T_t r | t \leq S)$ and $(V_t | t \leq S)$ satisfying

$$T_{s+t} r = T_s(T_t r) \quad (s+t \leq S) \quad (2.9)$$

$$T_t r = \int_0^t A(T_s r) ds \quad (t \leq S) \quad (2.10)$$

$$V_t = \int_0^t T_s r ds = \int_0^t [r + A V_s] ds \quad (t \leq S) \quad (2.11)$$

Proof The statements (2.7) and (2.8) follow from (2.3) as in [15] with $q^\delta(\dots) = q(\dots)$. For given r , the existence of a unique μ_x -bounded solution $\{V_t | t \leq S\}$ satisfying $V_t = \int_0^t [r + AV_s] ds$ is directly concluded from theorem 2.1 of [15] with $q^\delta(\cdot) = q(\cdot)$, $r^\delta(\cdot) = r(\cdot)$, $\Psi = 0$, $t = Z - t$ and $Z = S$ substituted. This same theorem also proves the existence of a unique μ_x -bounded family $\{T_t r | t \leq S\}$ satisfying (2.10) by substituting $r = 0$, $\Psi = r$ and $Z = t$. The semigroup-property (2.9) as well as the relation $V_t = \int_0^t T_s r ds$, finally, cannot be concluded directly from this theorem but in fact are implicitly proven by the proof of this theorem as based on discrete-time approximations, for which the discrete analogues of these statements are standard.

From standard Markov reward theory it is well-known in view of the above representation that the value $T_t r(i)$ and $V_t(i)$ represent the expected reward at time t respectively the expected total reward over time t when starting in state i at time 0 and incurring a reward r at time t respectively a reward rate r during $[0, t]$. The operator A is known as the infinitesimal operator or generator of the underlying jump process.

3 Approximate uniformization

Consider some fixed uniformization limit Q and truncation limit N and for any $i \leq N$, let $t[i] \leq N$ be some given state of truncation. Here we allow

$$N \leq \infty \quad \text{and} \quad Q \geq \sup_{i \leq N} q(i) \quad \text{or} \quad Q \leq \sup_{i \leq N} q(i)$$

Then for any $i \leq N$ we can define one-step transition probabilities

$$\bar{p}(i, j) = \begin{cases} 0 & j > N \\ [1 - [q(i)Q^{-1} \wedge 1]] & j = t[i] \\ [q(i)Q^{-1} \wedge 1] p(i, j) & j \neq i, j \neq t[i], j \leq N \\ [q(i)Q^{-1} \wedge 1] [p(i, t[i]) + \sum_{j > N} p(i, j)] & j = t[i] \neq i \\ [q(i)Q^{-1} \wedge 1] [p(i, i) + \sum_{j > N} p(i, j)] + [1 - [q(i)Q^{-1} \wedge 1]] & j = t[i] = i \end{cases} \quad (3.1)$$

The above construction has the following interpretation. For $N=\infty$ and $Q \geq \sup_i q(i)$, the second and third probability correspond exactly to the conditional transition probabilities under the standard uniformization approach (cf. [13], p.110) in which state dependent exponential holding times are uniformized to one and the same exponential holding time with parameter Q and dummy jumps introduced from a state into itself, as if the process is inspected at an exponential rate Q . For $N<\infty$, which is of computational interest, this uniformizing approach is retained but with truncated state space in which transitions exceeding the truncation limit are lumped together in some state $t[i]$. For $Q < \sup_{i \leq N} q(i)$ with either $N=\infty$ or $N<\infty$, as can be of interest both from a computational and theoretical point of view, the uniformization is relaxed to states with $q(i) \leq Q$ only. The special case $N<\infty$ and $Q \geq \max_i q(i)$ which is most practically interesting, will be referred to as "truncated uniformization".

Now let us investigate how the above construction for fixed Q and N can be exploited to "efficiently approximate" stationary performance measures of the continuous time Markov chain given in section 2. To this end, we adopt all notation from section 2 and with the reward rate r let functions $(\bar{V}_n | n=0,1,2,\dots)$ be defined by

$$\bar{V}_{n+1} = \sum_{k=0}^n \bar{T}_k r/Q = r/Q + \bar{T} \bar{V}_n \quad (3.2)$$

$$\bar{T}_{k+1} = \bar{T} \bar{T}_k \quad \text{where} \quad \bar{T}_0 = I \quad (3.3)$$

$$\bar{T} f(i) = \sum_j \bar{p}(i,j) f(j). \quad (3.4)$$

These reward functions thus correspond to the expected total reward over n periods of length $1/Q$ with one-step transition probabilities given by (3.1). Now let us assume that for some state ℓ :

$$\begin{aligned} \bar{g} &= \lim_{N \rightarrow \infty} \frac{Q}{N} \bar{V}_N(\ell) \\ \text{and} \quad g &= \lim_{t \rightarrow \infty} \frac{1}{t} V_t(\ell) \end{aligned} \quad (3.5)$$

exist, where V_t is the function given by (2.11), which represent the expected average reward per period and per unit of time of the discrete con-

struction (3.2) and the original continuous-time Markov chain respectively. The following key-theorem then shows that under appropriate conditions the approximate uniformization construction approximates the average reward, such as a performance measure, within an order $1/Q$.

Theorem 3.1 (Approximate uniformization) Suppose that for some B_1, B_2 :

$$\bar{T}_n \mu_2(\ell) \leq B_1 \quad (\forall n) \quad (3.6)$$

$$\|T_t \mu_r\|_{\mu_r} \leq B_2 \quad (\forall t) \quad (3.7)$$

and for some B_3 and all t :

$$\begin{aligned} & \left| \sum_{j \geq N} p(i, j) [V_t(j) - V_t(i)] \right| + \\ & \left| 1_{\{q(i) > Q\}} \sum_{j \leq N} p(i, j) [V_t(j) - V_t(i)] \right| + \\ & \left| [1 + 1_{\{q(i) > Q\}}] \sum_{j \leq N} p(i, j) [V_t(t[i]) - V_t(i)] \right| \leq \mu_q(i) \mu_r(i) B_3 / Q \end{aligned} \quad (3.8)$$

Then for all $t = n/Q$ and with $C = [K_q B_3 + (K+1) K_q K_r B_2]$:

$$|(\bar{V}_n - V_t)(\ell)| \leq t B_1 C / Q \quad (3.9)$$

$$|\bar{g} - g| \leq B_1 C / Q. \quad (3.10)$$

Proof Let $h=1/Q$, then by virtue of (3.2) and (2.11)

$$\bar{V}_n = h r + \bar{T} \bar{V}_{n-1} \quad (3.11)$$

$$V_{nh} = V_{nh-h} + \int_{nh-h}^{nh} (r + A V_s) ds.$$

Also noting that $\bar{T}$ is restricted to $\{1, \dots, N\}$, we thus conclude for $i \leq N$:

$$(V_{nh} - \bar{V}_n)(i) =$$

$$\begin{aligned}
 & V_{nh-h}(i) + h A V_{nh-h}(i) + \int_{nh-h}^{nh} A(V_s - V_{nh-h})(i) ds - \bar{T} \bar{V}_{n-1}(i) = \\
 & [A - (\bar{T}-I)h^{-1}]h V_{nh-h}(i) + \int_{nh-h}^{nh} A(V_s - V_{nh-h})(i) ds + \bar{T}(V_{nh-h} - \bar{V}_{n-1})(i)
 \end{aligned}
 \tag{3.12}$$

From (2.7), (3.1), (3.4) and (3.8), we conclude for arbitrary t and with $h^{-1}=Q$:

$$\begin{aligned}
 & |[A - (\bar{T}-I)h^{-1}]V_t(i)| = \\
 & |q(i) \sum_j p(i,j) \{V_t(j) - V_t(i)\} - [q(i) \wedge Q] * \\
 & \{ \sum_{j \leq N} p(i,j) [V_t(j) - V_t(i)] + \sum_{j > N} p(i,j) [V_t(t[i]) - V_t(i)] \}| = \\
 & |1_{\{q(i) > Q\}} [q(i) - Q] \sum_{j \leq N} p(i,j) [V_t(j) - V_t(i)] + \\
 & 1_{\{q(i) \leq Q\}} q(i) \sum_{j > N} p(i,j) [V_t(j) - V_t(t[i])] + \\
 & 1_{\{q(i) > Q\}} \{q(i) \sum_{j > N} p(i,j) [V_t(j) - V_t(i)] - \\
 & Q \sum_{j > N} p(i,j) [V_t(t[i]) - V_t(i)] \}| \leq \\
 & q(i) \mu_q(i) \mu_r(i) B_3 / Q \leq \\
 & \mu_2(i) K_q B_3 / Q.
 \end{aligned}
 \tag{3.13}$$

From (2.8), (2.10), (2.11), (3.7), $h=1/Q$ and the fact that $\|\cdot\|_{\mu_1} \leq \|\cdot\|_{\mu_r}$, we also conclude

$$\begin{aligned}
 & \|A(V_s - V_{nh-h})\|_{\mu_2} \leq (K+1)K_q \left\| \int_{nh-h}^s T_u r \, du \right\|_{\mu_1} \leq \\
 & (K+1)K_q \|r\|_{\mu_r} \left\| \int_{nh-h}^s T_u \mu_r \, du \right\|_{\mu_r} \leq (K+1)K_q K_r B_2 / Q.
 \end{aligned}
 \tag{3.14}$$

Further, note that $|f(i)| \leq \mu_2(i) \|f\|_{\mu_2}$ and $|\bar{T} f(i)| \leq \sum_j \bar{p}(i,j) |f(j)|$ for arbitrary function f . Hence, from substituting (3.13) and (3.14) in (3.12) after taking absolute values, we conclude that for any $i \leq N$ and with $C = [K_q B_3 + (K+1)K_q K_r B_2]$:

$$|(\bar{V}_n - V_{nh})(i)| \leq h \mu_2(i) [C/Q] + \sum_j \bar{p}(i,j) |(\bar{V}_{n-1} - V_{nh-h})(j)|. \quad (3.15)$$

As the probabilities $\bar{p}(i,j)$ remain restricted to states $j \leq N$, we thus find by iterating the latter inequality for $n, n-1, n-2, \dots, 0$ and substituting $V_0(\cdot) = \bar{V}_0(\cdot) = 0$:

$$|(\bar{V}_n - V_{nh})(i)| \leq h [C/Q] \sum_{k=0}^{n-1} (\bar{T}_k \mu_2)(i). \quad (3.16)$$

By applying condition (3.6) with $i=l$, the proof of inequality (3.9) is hereby completed. Inequality (3.10) immediately follows by the definitions (3.5). $\square$

Remark 3.2 (Condition 3.8). The right hand side of condition (3.8) may seem artificial by the inclusion of $\mu_q(i)$ and $1/Q$. However, for practical application one may typically think of the left hand side of (3.8) to be non-zero only for i such that $q(i)$ is approximately equal to or larger than Q . Roughly, it then comes down to verifying (3.8) with the right hand side replaced by $\mu_r(i) B_3$ in order to guarantee (3.10) with order of accuracy $1/Q$.

Remark 3.3 (Restricted/truncated uniformization). Below we particularize condition (3.8) for two special cases of interest.

(i) $N=\infty, Q<\infty$ (Restricted uniformization). As $N=\infty$ and only states with $K_q \mu_q(i) \geq q(i) \geq Q$ are involved in (3.8), in this case (3.8) reduces to:

$$||\sum_j \bar{p}(\cdot, j) [V_t(j) - V_t(\cdot)]||_{\mu_r} \leq B_3 K_q^{-1} \quad (3.17)$$

(ii) $N<\infty, Q \geq \max_{i \leq N} q(i)$ (Truncated uniformization). As $q(i) \leq Q$, condi-

tion (3.8) is now guaranteed by

$$\begin{aligned} & |\sum_{j>N} \bar{p}(i,j) [V_t(j) - V_t(i)]| + \\ & |\sum_{j>N} \bar{p}(i,j) [V_t(\bar{c}[i]) - V_t(i)]| \leq \mu_q(i) \mu_r(i) B_3 / Q. \end{aligned} \quad (3.18)$$

Particularly, with $t[i]=i$ for all i and $\mu_q(i) \geq (1-\delta)K_q^{-1}Q$ for all i such that $p(i,j)>0$ for some $j>N$, where $0\leq\delta<1$, this reduces to

$$|\sum_{j>N} p(i,j) [V_t(j)-V_t(i)]| \leq \mu_x(i) B_3 K_q^{-1}(1-\delta) \quad (3.19)$$

Remark 3.4 For the case $N=\infty$, the key-result (3.9) of theorem 3.1 resembles Theorem 2.2 of [15], showing that for this special case the construction 3.2 approximates the value V_t within an order $1/Q$ but with a conditioning constant C_t depending on t . From the proofs of this reference, however, this constant depending on t appears to be exponential in t (see the proof of Lemma 5.1). In contrast but most essentially, however, (3.9) proves that this constant can be reduced to a linear function in t so that the main result (3.10) can be concluded.

Remark 3.5 As (3.10) implies the order of accuracy $1/Q$ provided the conditions (3.6), (3.7) and (3.8) are satisfied, it also proves the approximate uniformization to be asymptotically exact in Q . Particularly, the truncated uniformization such as under (3.6), (3.7) and (3.19) seems to guarantee this asymptotic exactness. This will be illustrated in section 5.

4 Recursive verification of continuous-time conditions

The conditions (3.7) and (3.8) involve continuous-time functions that are to be estimated from above independently of t . This itself is a main problem as time-recursive techniques cannot be applied in continuous-time. This section therefore will show that the discrete-time recursive construction (3.2) and (3.3) can also be utilized to verify the conditions (3.7) and (3.8) in a discrete-time recursive manner.

To this end, approximation results will be established similar to (3.9) but without continuous-time conditions such as (3.7) or (3.8). In contrast with the right hand side of (3.9), however, the order of this approximation will grow exponentially in t so that a main result of interest as (3.10) cannot be concluded. But for fixed t , these approximations will suffice to verify (3.7) and (3.8).

Let $N < \infty$ and $Q \geq \max_{i \leq N} q(i)$ be arbitrary finite numbers and consider the one-step transition construction given by (3.1). Hence, with $h=1/Q$, we have

$$\bar{p}(i, j) = \begin{cases} 0 & j > N \\ 1-h q(i) & j = i \neq t[i] \\ h q(i, j) & j \neq i, j \neq t[i] \\ h[q(i, t[i]) + \sum_{j > N} q(i, j)] & j = t[i] \neq i \\ h[q(i, i) + \sum_{j > N} q(i, j)] + [1-h q(i)] & j = t[i] = i \end{cases} \quad (4.1)$$

Let $\bar{T}$, $\bar{T}_k$ be defined by (3.3) and (3.4) and write $\bar{A} = [\bar{T} - I]h^{-1}$. The following key-lemma is then obtained as a truncated analog of lemma 4.1 from [15]. Its proof essentially relies upon the technical assumptions (2.4) or (2.5).

Lemma 4.1 For $k=0,1,2$ and some constant C independent of N and Q :

$$(\bar{T}\mu_k)(i) \leq (1+hC) \mu_k(i) \quad (i \leq N) \quad (4.2)$$

Proof Noting that $\mu_k(t[i]) \leq \mu_k(j)$ for all $j > N$ and with $q(i) = \sum_j q(i, j)$ we find

$$\begin{aligned} \bar{T}\mu_k(i) &= \mu_k(i) + h \sum_{j \leq N} q(i, j) \mu_k(j) \\ &\quad - h q(i) \mu_k(i) + h \mu_k(t[i]) \sum_{j > N} q(i, j) \\ &\leq \mu_k(i) + h \sum_j q(i, j) [\mu_k(j) - \mu_k(i)] \end{aligned} \quad (4.3)$$

The proof is hereby completed under assumption (2.5), while under (2.4) the proof is guaranteed by noting that $\mu_k(j) \leq \mu_k(i)$ for $j \leq i$ so that the latter inequality can be reduced to

$$\bar{T}\mu_k(i) \leq \mu_k(i) + h \sum_{j > i} q(i, j) [\mu_k(j) - \mu_k(i)] \quad (4.4)$$

□

The following lemma is a minor modification of lemma 2.1. Its proof follows directly from theorem 2.1 of [15] with $q^\delta(\cdot)=q(\cdot)$, $r^\delta(\cdot)=r(\cdot)$, $t=Z-t$ and $Z=S$.

Lemma 4.2 For arbitrary finite S and μ_k -bounded functions r and ψ there exists a unique μ_r -bounded family $\{W_t | t \leq S\}$ satisfying

$$W_t = \psi + \int_0^t [r + A W] ds. \quad (4.5)$$

Further,

$$W_t = \int_0^t T_s r ds + T_t \psi \quad (4.6)$$

Now consider some fixed t and μ_k -bounded functions r and ψ . Then with $\ell = [t/h]$, the functions $\{\bar{W}_n | n=0,1,2,\dots,\ell\}$ can be defined by

$$\begin{cases} \bar{W}_{n+1} = h r + \bar{T}_n \\ \bar{W}_\ell = \psi. \end{cases} \quad (n < \ell) \quad (4.7)$$

The following lemma then relates this special recursive construction to the continuous-time formulation (4.5). Its proof is similar to that of theorem 3.1 but with some minor though essential technical differences as the conditions (3.7) and (3.3) are excluded.

Lemma 4.3 For given N and $Q \geq \max_{1 \leq N} q(i)$ assume that for some constant $r > 0$:

$$\mu_q(i) \geq rQ \quad \text{if} \quad \sum_{j > N} p(i,j) > 0 \quad (i \leq N) \quad (4.8)$$

Then for arbitrary $S < \infty$, any $t \leq S$, some constant C_S independent of N and Q , $\ell = [t/h]$ and arbitrary μ_r -bounded function f , we have

$$|(\bar{T}_\ell - T_t)f(i)| \leq \mu_2(i)C_t/Q \quad (i \leq N). \quad (4.9)$$

$$|(\bar{W}_\ell - W_t)(i)| \leq \mu_2(i)C_t/Q \quad (i \leq N). \quad (4.10)$$

Proof We will first prove (4.10) for arbitrary μ_r -bounded functions r and ψ . From (4.5) and (4.3), we obtain similarly to (3.11) and (3.12) that for any $nh \leq S$ and $i \leq N$:

$$\begin{aligned} (W_{nh} - \bar{W}_n)(i) = \\ [A - (\bar{T} - I)h^{-1}]h W_{nh-h}(i) + \\ \int_{nh-h}^{nh} A(W_s - W_{nh-h})(i) ds + \bar{T}(W_{nh-h} - \bar{W}_{n-1})(i) \end{aligned} \quad (4.11)$$

Now noting that $q(i) \leq Q$ for all $i \leq N$, for all $s \leq S$ we obtain as in (3.13):

$$|[A - (\bar{T} - I)h^{-1}]W_s(i)| = q(i) |\sum_{j>N} p(i, j) [W_s(j) - W_s(t[i])]| \quad (4.12)$$

As W_s is μ_r -bounded while also $\mu_r(t[i]) \leq \mu_r(j)$ for all $j > N$, by virtue of condition (2.3) with $k=0$ and the technical condition (4.8), the latter expression can be estimated from above by

$$\begin{aligned} 2 q(i) K \mu_r(i) \|W_s\|_{\mu_r} \leq \\ 2 K K_q \mu_2(i) \|W_s\|_{\mu_r} / \mu_q(i) \leq \mu_2(i) C_1 / Q \end{aligned} \quad (4.13)$$

where $C_1 = 2KK_q \max_{s \leq S} \|W_s\|_{\mu_r} / \tau$. Further, from (2.8.) and (4.5):

$$\begin{aligned} \|A(W_s - W_{nh-h})\|_{\mu_r} \leq \\ (K+1)K_q \left\| \int_{nh-h}^s [\tau + AW_u] du \right\|_{\mu_1} \leq \\ h(K+1)K_q [\|r\|_{\mu_1} + (K+1)K_q \sup_{u \leq S} \|W_u\|_{\mu_r}] \leq hC_2 = C_2/Q. \end{aligned} \quad (4.14)$$

for a constant C_2 independent of N and $Q=1/h$. Then similarly to (3.15), by substituting (4.13) and (4.14) in (4.11) after taking absolute values, we find for any $i \leq N$:

$$|(\bar{W}_n - W_{nh})(i)| \leq h \mu_2(i) [C_3/Q] + |\bar{T}(\bar{W}_{n-1} - W_{nh})(i)| \quad (4.15)$$

where $C_3 = C_1 + C_2$ does not depend on N and $Q = 1/h$. Hence, by lemma 4.1 and $(1+hC) \leq \exp(hC)$, we have for all $nh \leq S$:

$$\begin{aligned} \|\bar{W}_n - W_{nh}\|_{\mu_2} &\leq h[C_3/Q] + (1+hC) \|\bar{W}_{n-1} - W_{nh}\|_{\mu_2} \\ &\leq \dots \leq [nh](1+hC)^n [C_3/Q] \\ &\leq S e^{CS} [C_3/Q]. \end{aligned} \quad (4.16)$$

Finally, with $n = [t/h]$ and $h = 1/Q$, we conclude as in (4.14):

$$\|W_{nh} - W_t\|_{\mu_2} \leq C_2/Q \quad (4.17)$$

By combining (4.16) and (4.17), inequality (4.10) is hereby proven for arbitrary μ_r -bounded functions r and ψ . By letting r be identical to 0 and substituting $\psi = f$, (4.9) immediately follows from (4.6) and (4.10). $\square$

Theorem 4.4 (Discrete verification) For some $\epsilon > 0$ and sequences $\{N_m\}_{m=1,2,\dots}$ and $\{Q_m\}_{m=1,2,\dots}$ with $N_m \rightarrow \infty$ and $Q_m \rightarrow \infty$ as $m \rightarrow \infty$, suppose that for all $m = 1, 2, \dots$ condition (4.8) of lemma 4.3 is satisfied with $N = N_m$ and $Q = Q_m$. Further, assume that for all m , some constants B and $B(k, j)$ for given k, j :

$$\bar{T}_n \mu_r(i) \leq \mu_r(i) B \quad (i \leq N) \quad (4.18)$$

and

$$|\bar{V}_n(j) - \bar{V}_n(k)| \leq B(k, j) \quad (n \geq 0) \quad (4.19)$$

with $\bar{T}_n$ and $\bar{V}_n$ as per (3.2) and (3.4) based on (4.1) with $N = N_m$ and $Q = Q_m$, and provided $k, j \leq N_m$. Then also for all i and that given k, j :

$$T_t \mu_r(i) \leq \mu_r(i) B \quad (t \geq 0) \quad (4.20)$$

$$|V_t(j) - V_t(k)| \leq B(k, j) \quad (t \geq 0) \quad (4.21)$$

Proof By letting $N = N_m \rightarrow \infty$ and $Q = Q_m \rightarrow \infty$ as $m \rightarrow \infty$, this follows directly from lemma 4.3. $\square$

Remark 4.5 (Monotonicity properties) Theorem 4.4 enables us to verify the conditions (3.7) and (3.8) in an inductive manner based on the recursive constructions (3.1), (3.2) and (3.3). As will be illustrated in detail in the next section, verification of the conditions (3.7) and (3.8) can hereby essentially be reduced to showing that the one-step transition structure preserves monotonicity properties (see lemma 5.1, lemma 5.2 and lemma 5.4).

5 Application: Queueing systems with breakdowns

To illustrate how the approximate uniformization technique applies and how the necessary conditions can be verified in practical situations, this section concerns an application to queueing systems with breakdowns. Such systems do not exhibit a product form expression for the steady state distribution and thus require a numerical computation. Moreover, in this case the necessary conditions such as (3.7) and (3.8) cannot be verified directly by birth-death type arguments as a two- and thus essentially more-dimensional state space is involved. To this end, monotonicity results are to be proved. Such results are currently of interest in the literature (cf. [9], [10], [11], [12], [13] and [19]). The present application, however, requires special technicalities as infinite intensities are involved. This has not been touched upon earlier in the literature. The proof techniques therefore are of general interest.

For the purpose of illustration and exposition, we restrict our attention to the truncated uniformization case (see remark 3.3 ii) as this case is the most natural one from a numerical point of view. However, as the essential technicalities such as (3.7) and (3.8) are hereby dealt with, similar results can be expected for other approximate uniformizations such as restricted uniformization (see remark 3.3 i).

As per inequality (3.10) of theorem 3.1, the accuracy of the truncated uniformization proposed will be shown to be of order $1/Q$ with an explicit conditioning constant B_3C . As performance measure of interest we will consider the system throughput.

5.1 Model

Consider an infinite server system that is subject to breakdowns independently of whether customers are present or not at an exponential rate with parameter ν_1 . A breakdown renders the system inoperative for an exponential period with parameter ν_0 . The service requirements are assumed to be exponential with parameter μ . The interarrival times are also exponential but with state dependent parameter $\lambda(p, \theta)$ whenever the system is in state (p, θ) denoting that p customers are present while the system is in status θ , where $\theta=1$ means operative (up) and $\theta=0$ means inoperative (down). Here we assume that for some $\lambda < \infty$ and all (p, θ) :

$$\begin{aligned}\lambda(p, 1) &\geq \lambda(p, 0) \\ \lambda &\geq \lambda(p, \theta) \geq \lambda(p+1, \theta).\end{aligned}\tag{5.1}$$

For example, we can have $\lambda(n, \theta)=0$ whenever $\theta=0$ and $n \geq Z$ for some Z , while $\lambda(n, \theta)=\lambda$ otherwise, representing a system in which arrivals are blocked above a certain level Z when the system is down.

The description above is known in the literature as the "independent" breakdown case in contrast with the so-called "active" breakdown case in which the system can go down only when it is serving (cf. [4], p.101). Despite intuition, either case is essentially as equally untractable analytically. For constant $\lambda(.,.)=\lambda$ and a single instead of infinite server, closed form expressions have been obtained for the generating function of the queue length (cf. [4], p.103). Similar expressions can be expected for the infinite server case but even then explicit expressions for the steady state distribution do not seem obtainable. A product form expression can be shown to hold (cf. [16]) only when $\lambda(., \theta)=0$ for $\theta=0$. For the general case numerical computation is thus required. As infinite transition rates are involved, truncated uniformization will be proposed. As performance measure of interest we investigate the throughput, where it is noted that this measure is known explicitly only when the arrival rate is constant.

5.2 Truncated uniformization

The transition rates $q([p,\theta],[p',\theta'])$ for a transition from state $[p,\theta]$ into $[p',\theta']$ are given by

$$\begin{cases} \lambda(p,\theta) & \text{for } [p',\theta'] = [p+1,\theta] \\ p \mu \mathbf{1}_{\{\theta=1\}} & \text{for } [p',\theta'] = [p-1,1] \\ \nu_0 \mathbf{1}_{\{\theta=0\}} & \text{for } [p',\theta'] = [p,1] \\ \nu_1 \mathbf{1}_{\{\theta=1\}} & \text{for } [p',\theta'] = [p,0] \end{cases} \quad (5.2)$$

Further, let

$$r(p,\theta) = p \mu \mathbf{1}_{\{\theta=1\}} \quad (5.3)$$

which represents the departure rate whenever the system is operative with p customers. First, let us investigate the conditions (2.1). Clearly (2.1) and (2.2) are satisfied with

$$\begin{aligned} \mu_r(p,\theta) &= (1+p), & K_r &= \mu \\ \mu_q(p,\theta) &= (1+p), & K_q &= \lambda + \mu + \nu_0 + \nu_1. \end{aligned}$$

Further, with the state labeling given below one directly verifies the conditions (2.6) which in turn, as per reference [1], guarantees conditions (2.3) and (2.4) with $K=8$. Lemmas 2.1 and 2.2 thus apply. Particularly, the functions V_t are well-defined and their limits as per (3.5) necessarily will equal the unique throughput g . The approximate uniformization theorem 3.1 thus becomes of interest.

To this end, label the states $[p,\theta]$ by the number $2p+\theta+2 \mathbf{1}_{\{p>0\}}$ and throughout this section, let i and j denote the label corresponding to a state $[p,\theta]$ and $[p',\theta']$ respectively. Further, the preceding notation will be adopted either with i and j representing labels or with i and j replaced by their corresponding two-dimensional states. For example, we write both $q(i,j)$ and $q([p,\theta],[p',\theta'])$ for the transition rates as given above.

Let the operators $\{T_t | t \geq 0\}$ and the functions $\{V_t | t \geq 0\}$ be given by (2.10) and (2.11) for the above transition rates and reward structure. Then V_t represents the expected total number of departures over a time period of length t , so that the system throughput is determined by

$$g = \lim_{t \rightarrow \infty} \frac{1}{t} V_t \quad (0, 1].$$

In order to apply theorem 3.1 we first wish to establish the conditions (4.18) and (4.19) for concluding the estimates (3.7) and (3.8). These estimates are the most essential step for practically applying theorem 3.1.

To this end, consider any arbitrary truncation limit L on the queue size and let the transition probabilities $\bar{p}([p, \theta], [p', \theta'])$ be defined by (4.1) with $N = 2L+3$ and $\tau([L, \theta]) = [L, \theta]$. Then with

$$Q = (L\mu + \lambda + \nu_0 + \nu_1 + \mu) \geq \max_{\{[p, \theta] | p \leq L, \theta=0,1\}} \{q([p, \theta]) + \mu\}, \quad (5.4)$$

we have

$$\bar{p}([p, \theta], [p', \theta']) = \begin{cases} 1 - q([L, \theta])/Q + \lambda([L, \theta])/Q & [p', \theta'] = [p, \theta] = [L, \theta] \\ 1 - q([L, \theta])/Q & [p', \theta'] = [p, \theta] \neq [L, \theta] \\ q([p, \theta], [p', \theta'])/Q & \text{otherwise} \end{cases} \quad (5.5)$$

for all $[p, \theta]$ with $p \leq L$. In words, this construction rejects arrivals exceeding the threshold L . First let us prove that this construction preserves monotonicity.

Lemma 5.1 For any function $f(n, \theta)$ such that

$$f(p+1, \theta) - f(p, \theta) \geq 0 \quad (p \geq 0, \theta=0,1) \quad (5.6)$$

also

$$\bar{T} f(p+1, \theta) - \bar{T} f(p, \theta) \geq 0 \quad (p \geq 0, \theta=0,1). \quad (5.7)$$

Proof For any $p+1 \leq L$ and $\theta=0,1$, we obtain from (5.1) and (5.5):

$$\begin{aligned}
 \bar{T} f(p+1, \theta) - \bar{T} f(p, \theta) = & \\
 & \{ [\lambda(p+1, \theta)/Q] 1_{\{p+1 < L\}} f(p+2, \theta) + 1_{\{p+1=L\}} f(L, \theta) \} \\
 & + [(\mu + 1) 1_{\{\theta=1\}}/Q] f(p, \theta) + [\nu_0 1_{\{\theta=0\}}/Q] f(p+1, 1) \\
 & + [\nu_1 1_{\{\theta=1\}}/Q] f(p+1, \theta) \\
 & + (1 - [\lambda(p+1, \theta) + (\mu + 1) 1_{\{\theta=1\}} + \nu_\theta]/Q) f(p+1, \theta) \\
 & - ([\lambda(p, \theta)/Q] f(p+1, \theta) \\
 & + [p \mu 1_{\{\theta=1\}} 1_{\{p > 0\}}/Q] f(p-1, 1) \\
 & + [\nu_1 1_{\{\theta=1\}}/Q] f(p, 0) + [\nu_0 1_{\{\theta=0\}}/Q] f(p, 1) \\
 & + (1 - [\lambda(p, \theta) + p \mu 1_{\{\theta=1\}} 1_{\{p > 0\}} + \nu_\theta]/Q) f(p, \theta)) \\
 = & [\lambda(p, \theta)/Q] 1_{\{p+1 < L\}} [f(p+2, \theta) - f(p+1, \theta)] \\
 & + [\lambda(p, \theta)/Q] 1_{\{p+1=L\}} [f(L, \theta) - f(L, \theta)] \\
 & + [(\lambda(p, \theta) - \lambda(p+1, \theta))/Q] [f(p+1, \theta) - f(p+1, \theta)] \\
 & + [p \mu 1_{\{\theta=1\}} 1_{\{p > 0\}}/Q] [f(p, \theta) - f(p-1, \theta)] \\
 & + [\mu 1_{\{\theta=1\}}/Q] [f(p, \theta) - f(p, \theta)] \\
 & + [\nu_1 1_{\{\theta=1\}}/Q] [f(p+1, 0) - f(p, 0)] \\
 & + [\nu_0 1_{\{\theta=0\}}/Q] [f(p+1, 1) - f(p, 1)] \\
 & + [1 - (\lambda(p, \theta) + (\mu + 1) 1_{\{\theta=1\}} + \nu_\theta)/Q] [f(p+1, \theta) - f(p, \theta)].
 \end{aligned}$$

Noting that the second, third and fifth term of the last expression are equal to 0, applying (5.6) to the other terms, and recalling (5.4), we have hereby completed the proof. $\square$

Lemma 5.2 For $k=0,1,2$ and all $n \geq 0$:

$$\bar{T}_{n+1} \mu_k(0,1) \leq \bar{T}_n \mu_k(0,1) \quad (5.8)$$

Proof By induction to n we will show that for any function f satisfying (5.6), we also have

$$(\bar{T}_{n+1}f - \bar{T}_nf)(0,1) \geq 0 \quad (5.9)$$

For $n=0$ this follows by:

$$\begin{aligned} \bar{T}f(0,1) &= [\lambda(0,1)/Q] f(1,1) \\ &+ (1 - [\lambda(0,1)/Q]) f(0,1) \geq f(0,1). \end{aligned}$$

Suppose that (5.9) holds for $n \leq m-1$, then for $n=m$:

$$(\bar{T}_{m+1}f - \bar{T}_mf)(0,1) = (\bar{T}_m - \bar{T}_{m-1})(\bar{T}f)(0,1) \geq 0,$$

by virtue of the induction hypothesis provided $(\bar{T}f)$ satisfies (5.6). This, however, is proven by lemma 5.1. $\square$

Lemma 5.3 Let $\rho = \lambda[\mu^{-1} + \nu_0^{-1} + \mu^{-1}\nu_1\nu_0^{-1}]$. Then, for $k=0,1,2$

$$\bar{T}_n \mu_k(0,1) \leq e^{-\rho} \sum_{j=1}^{\infty} (1+j)^{(k+1)} \rho^j / j! = B(k). \quad (5.10)$$

Proof The arrival rate of the original queueing system is estimated from above by λ . Also, as the mean number of breakdowns during a service is estimated from above by ν_1/μ while upon arrival the system can be down, the mean sojourn time of a customer in the original queueing system is estimated from above by $[1 + \nu_1/\mu]\nu_0^{-1} + \mu^{-1}$. Therefore, with X_B the queue length of

the original breakdown system in equilibrium and X the queue length of a stationary $M|M|\infty$ system with arrival rate λ and mean service time $[1+\nu_1/\nu_0]/\mu$, one can prove by stochastic monotonicity results such as in [9], [10], [11], [18] and [19] that

$$E(1+X_B)^k \leq E(1+X)^k = C(k).$$

Now consider the original breakdown system with the modification that arrivals exceeding the threshold L are rejected and let X_B^L be the queue length of this system in equilibrium. Then, as intuitively clear, one can prove as in [16] that also

$$E(1+X_B^L)^k \leq E(1+X_B)^k.$$

On the other hand, by virtue of the construction (5.5) and the standard uniformization technique (cf. [15], p.110), the steady state distributions of this truncated breakdown model and the construction (5.5) are the same, so that particularly

$$\lim_{n \rightarrow \infty} \bar{T}_n \mu_k = E(1+X_B^L)^k.$$

The monotonicity as proven by lemma 5.2 thus completes the proof. $\square$

The following lemma is concerned with estimating the bias terms of the reward structure of interest, being the throughput. To this end, let

$$r([p, \theta]) = p \mu 1_{\{\theta=1\}} \quad (5.11)$$

and consider the functions $\bar{V}_n$ as given by (3.2) as based upon $\bar{T}$ given by (3.4) and (5.5).

Lemma 5.4 For all r , $p+1 \leq L$ and $\theta=0,1$ we have

$$0 \leq \bar{V}_n([p+1, \theta]) - \bar{V}_n([p, \theta]) \leq 1 \quad (5.12)$$

Proof Since for $p+1 \leq L$:

$$\begin{aligned} & \bar{V}_{m+1}([p+1, \theta]) - \bar{V}_{m+1}([p, \theta]) = \\ & [\mu 1_{\{\theta=1\}}/Q] + \bar{T} \bar{V}_m([p+1, \theta]) - \bar{T} \bar{V}_m([p, \theta]), \end{aligned}$$

as in the proof of lemma 4.1 with f replaced by $\bar{V}_m$ and terms equal to zero omitted, we find

$$\begin{aligned} & \bar{V}_{m+1}([p+1, \theta]) - \bar{V}_{m+1}([p, \theta]) = \\ & [\mu 1_{\{\theta=1\}}/Q] \\ & + [\lambda(p, \theta)/Q] 1_{\{p+1 < L\}} [\bar{V}_m(p+1, \theta) - \bar{V}_m([p, \theta])] \\ & + [p \mu 1_{\{\theta=1\}} 1_{\{p > 0\}}/Q] [\bar{V}_m([p, 1]) - \bar{V}_m([p-1, 1])] \\ & + [\nu_1 1_{\{\theta=1\}}/Q] [\bar{V}_1([p+1, 0]) - \bar{V}_m([p, 0])] \\ & + [\nu_0 1_{\{\theta=0\}}/Q] [\bar{V}_1([p+1, 1]) - \bar{V}_m([p, 0])] \\ & + [1 - (\lambda(p, \theta) + (p+1) \mu 1_{\{\theta=1\}} + \nu_\theta)/Q] [\bar{V}_m(p+1, \theta) - \bar{V}_m(p, \theta)] \quad (5.13) \end{aligned}$$

We can now give the proof by induction to n . Clearly, (5.12) holds for $n=0$ as $\bar{V}_0(\dots)=0$. Suppose that (5.12) holds for $n \leq m$, then substitution of the lower estimate 0 in (5.13) directly yields the lower estimate 0 also for $n=m+1$. To conclude also the upper estimate 1 for $n=m+1$, note that the first term of the right hand side of (5.13) is compensated by $\mu 1_{\{\theta=1\}}$ of the last term together with the upper estimate 1 for $n=m$ by induction hypothesis and recall (5.4). $\square$

We are now able to apply theorems 3.1 and 4.3. To this end, let $\bar{g}$ be defined by (3.2) and (3.5) as based upon the truncated uniformization (5.5) for given truncation limit L .

Theorem 5.5 Let $B(0)$ and $B(2)$ be given by (5.10). Then for any queue size truncation limit L and $Q = [L\mu + \nu_0 + \nu_1 + \lambda + \mu]$:

$$|\bar{g} - g| \leq B(2)C/Q = O(1/L) \quad (5.14)$$

where

$$C = [K_q B_3 + 9 K_q K_r B(0)]$$

$$K_q = [\lambda + 2\mu + \nu_0 + \nu_1]$$

$$B_3 = K_q/(L+1)$$

$$K_r = \mu.$$

Proof First note that the label $N = 2L+3$ corresponds to the state $[L,1]$. The only states with label i such that a state with label $j > N$ can be reached, therefore, are the states $[L,\theta]$ with $\theta=0$ or $\theta=1$. For these states one has

$$\mu_q([L,\theta]) = (1+L) \geq \tau Q \quad (5.15)$$

for $\tau^{-1} = [\lambda + 2\mu + \nu_0 + \nu_1]$. Condition (4.8) is thus satisfied with fixed $\tau > 0$ for any $N = 2L+1$, $Q = [L\mu + \nu_0 + \nu_1 + \lambda + \mu]$ and arbitrary L . By letting $L \rightarrow \infty$ and recalling lemma 5.3 for $k=0$ and lemma 5.4, the conditions of theorem 4.4 and thus inequalities (4.20) and (4.21) are satisfied with

$$B = B(0) \text{ given by (5.10)} \quad (5.16)$$

$$B([L+1,\theta], [L,\theta]) = 1 \quad (5.17)$$

for arbitrary L . Now consider a fixed L . Then by virtue of lemma 5.3 and (5.16) we have thus verified the conditions (3.6) and (3.7) of theorem 3.1 with $\ell=1$ the label of state $[0,0]$ and

$$B_1 = B(2),$$

$$B_2 = B(0)$$

as per lemma 5.3. Further, as $t[L,\theta] = [L,\theta]$ and $q([p,\theta]) \leq Q$ for all

$p \leq L$ and $\theta=0,1$, by using (5.17) as estimate for $|V_t([L+1,\theta]) - V_t([L,\theta])|$ and substituting $\mu_q([L,\theta]) = \mu_r([L,\theta]) = (L+1)$, one immediately verifies (3.8) with

$$B_3 = Q/(L+1)^2 \leq [1+2\mu+\nu_0+\nu_1]/(L+1).$$

Application of theorem 3.1, where it is noted that $K=8$ as by (2.3), completes the proof. $\square$

References

- [1] Gihman, I.I. and Skorohod, A.V. (1979), Controlled stochastic processes, Springer-Verlag, Berlin.
- [2] Jensen, A. (1953), Markov Chains as an Aid in the Study of Markov Processes, Skan. Aktuarietidskr. 36, 87-91.
- [3] Kemeny, J.G., Snell, J.L. and Knapp, A.W. (1966), Denumerable Markov Chains, Van Nostrand, Princeton, N.J.
- [4] Jaiswal, N.K. (1968), Priority queues, Academic Press, New York.
- [5] Pliska, S.R. (1975), Controlled jump processes, Stochastic Prod.Appl. 3, 259-282.
- [6] Ross, S.M. (1981), Approximating transition probabilities and mean occupation times in continuous-time Markov chains, Prob. Eng. Inf. Sc. Vol.1, 251-264.
- [7] Seneta, E. (1980), Non-negative matrices and Markov chains, Springer-Verlag.
- [8] Serfozo, R.F. (1979), An Equivalence between Continuous and Discrete Time Markov Decision Processes, Oper.Res., 27, 616-620.
- [9] Shanthikumar, J.G. and Yao, D.D. (1987), General Queueing Networks: Representation and Stochastic Monotonicity, Research Report, University of California, Berkeley.
- [10] Shanthikumar, J.G. and Yao, D.D. (1987), Stochastic Monotonicity of the Queue Lengths in Closed Queueing Networks, Oper.Res., 35, 583-588.
- [11] Stoyan, D. (1983), Comparison Methods for Queues and Other Stochastic Models, Wiley, New York.

- [12] Suri, R. (1985), A Concept of Monotonicity and Its Characterization for Closed Queueing Networks, Oper.Res., 33, 606-624.
- [13] Tijms, H.C. (1986), Stochastic Modelling and Analysis; A computational Approach, Wiley, New York.
- [14] Tijms, H.C. and Mikeboom, A.M. (1986), A simple technique in Markovian control with applications to resource allocation, Oper.Res.Letters, 1, 25-32.
- [15] Van Dijk, N.M. (1988), On the finite horizon Bellman equation for controlled Markov jump models with unbounded characteristics, Stochastic Procc.Appl. 28, 141-157.
- [16] Van Dijk, N.M. (1988), Simple Bounds for Queueing Systems with Break-downs, Performance Evaluation, 8, 117-128.
- [17] Van Dijk, N.M. (1988), Truncation of Markov chains with applications to queueing, Research Report, Free University, Amsterdam.
- [18] Van Dijk, N.M. and Van der Wal, J. (1987), Simple bounds and monotonicity results for multi-server exponential tandem queues. To appear: Queueing Systems.
- [19] Whitt, W. (1981), Comparing counting processes and queues, Adv.Appl. Prob. 13, 207-220.
- [20] Yoon, B.S. and Shanthikumar, B.S. (1988), Bounds and approximations for the transient behaviour of continuous time Markov chains, Research Report, University of California, Berkeley.