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INTERCONNECTED METROPOLITAN AREA NETWORKS

Nico M. van Dijk

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A Note on Product Forms for
Interconnected Metropolitan Area Networks

Nico M. van Dijk
Free University, Amsterdam, The Netherlands

Abstract It is shown that a recently derived product form result in [4] for two interconnected metropolitan area networks (MAN's) directly extends to both more complex structures and non-exponential transmission times by a simple result adopted from [1].



1 Introduction

In [1] a framework has been developed so as to identify and characterize interconnection networks such as multi-access communication systems that have a product form. To this end, an invariance condition in terms of concrete system protocols such as blocking was provided.

This note aims to indicate that this condition directly includes and extends a recently derived product form result (cf. [4]) for interconnected metropolitan area networks (MAN's). For the purpose of self-containedness and illustration, the presentation is restricted to a simple case, which has also been studied earlier in the literature under exponential assumptions (cf. [2],[3]). A self-contained proof for this case is given which extends the product form result from [4] to more complex structures and non-exponential transmission times. Some novel examples are given.

2 Model and Product Form

Consider a system of N components, numbered $1, \dots, N$. Each component is alternatively in an idle or active mode for random amounts of time as according to a think and holding time distribution function F^h and G^h respectively for component $h, h=1, \dots, M$ and the access protocol described below. When component h completes a think time, during which it is called idle, while from the other components $h_1, \dots, h_n$ are active, component h will start a holding time, during which it is called active, provided

$$\{h_1, \dots, h_n\} \cup h \in C$$

where C is some set of states such that

$$\{h_1, \dots, h_n\} \in C \Rightarrow \{h_1, \dots, h_{j-1}, h_{j+1}, \dots, h_n\} \in C \quad (j=1, \dots, n).$$

Otherwise, component h has to restart a new think time and thus remains idle. When a component completes a holding time, it always becomes idle and starts a new think time.

Throughout, let a state $H = \{h_1, \dots, h_n\}$ denote that components $h_1, \dots, h_n$ are active (say in increasing order), and denote by $H \pm h$ the state in which component h is added to (+ sign) or deleted from (- sign) H as an active component.

Without loss of generality assume that the think and holding time distribution F_h and G_h respectively have continuous density functions $f_h(\cdot)$ and $g_h(\cdot)$ with means σ_h and τ_h respectively. Let the state

$$(S, T) = ((s_1, t_1), \dots, (s_N, t_N))$$

denote that component i is in mode s_i , where $s_i=1$ stands for idle and $s_i=2$ for active, with a residual time t_i up to completion of the current think time when $s_i=1$ or holding time when $s_i=2$. For a given specification $S = (s_1, \dots, s_N)$, let H denote the corresponding active components and write $\pi((S, T))$ and $\pi(H)$ for the stationary density and stationary probability of states (S, T) and H respectively. The following two theorems will be proven. The first is the key theorem. The second is the more practical consequence.

Theorem 2.1 With c a normalizing constant, we have

$$\pi((S, T)) = c \prod_{\{h: s_h=1\}} [1-F_h(t_h)] \prod_{\{h: s_h=2\}} [1-G_h(t_h)] \quad (2.1)$$

Proof We need to verify the global balance or forward Kolmogorov equations assuming without loss of generality that these have a unique solution. To this end, for a given state (S, T) and component i , denote by

$$(S, T) = (s_i, t_i)_i + (\hat{s}_i, \hat{t}_i)_i$$

the same state with the specification for component i changed from (s_i, t_i) in $(\hat{s}_i, \hat{t}_i)$. Further, we write 0^+ for a right hand limit at 0 and $1(A)$ for the indicator of an event A , i.e. $1(A)=1$ if A is satisfied and 0 otherwise. Then, for a fixed state (S, T) with active components represented by $H \in C$, the global balance equations become:

$$\sum_{\{h:s_h=1\}} \left\{ \frac{\partial}{\partial t_h} \pi((S,T)) + \pi((S,T)) g_h(t_h) 1(H+h \notin C) \right. \\ \left. + P((S,T)) - (1, t_h)_h + (2, 0^+)_h g_h(t_h) 1(H+h \in C) \right\} + \\ \sum_{\{h:s_h=2\}} \left\{ \frac{\partial}{\partial t_h} \pi((S,T)) + \pi((S,T) - (2, t_h)_h + (1, 0^+)_h) f_h(t_h) \right\} = 0. \quad (2.2)$$

Noting that $\partial/\partial t F_h = f_h$ and $\partial/\partial t G_h = g_h$, substitution of (2.1) in (2.2) directly shows that for each h separately the term within braces (...) is equal to 0, which completes the proof. $\square$

Theorem 2.2 With $\tilde{c} = c(\sigma_1) \dots (\sigma_N)$ a normalizing constant, we have

$$\pi(H) = \tilde{c} \prod_{h \in H} [\tau_h / \sigma_h].$$

Proof This follows directly from expression (2.1) by renormalizing after dividing by $\sigma_2 \sigma_2 \dots \sigma_N$ and integrating over all possible residual times t_h where it is to be noted that

$$\int_0^\infty [1 - F_h(t)] dt = \sigma_h, \quad \int_0^\infty [1 - G_h(t)] dt = \tau_h. \quad \square$$

Remark 2.3 Note that the blocking protocol of restarting a new think time corresponds to the blocked-calls-cleared protocol used in [4].

Remark 2.4 In correspondence with [2] and [3], the blocking mechanism for the components to become active can be called "coordinate convex". In these references exponentiality assumptions are made.

Remark 2.5 Note that the proof of theorem 2.1 is based on showing "balance" per individual component separately. Notions of partial balance are known to be responsible for insensitive product form expressions.

Remark 2.6 In [1] a more general mechanism is considered which also leads to a product form expression provided a so-called invariance condition is guaranteed. This includes for instance examples with priorities or randomized blocking. This paper will restrict to an application to MAN-systems in the next section.

3 Application: Interconnected Metropolitan Area Networks (MAN's)

Consider a communication system with two groups of subscribers, say a group A and B with M and N subscribers, such as representing two metropolitan or local area networks. Both within a group and in between the groups communication between subscribers might be possible. To this end, number all subscribers $1, \dots, M+N$ and identify each possible connection from a source subscriber m to a destination subscriber n as a component (m,n) . Let $1/\lambda(m,n)$ and $1/\mu(m,n)$ be the associated mean scheduled transmission and calling time, provided transmissions over this connection will take place (i.e., provided $\lambda(m,n)$ and $\mu(m,n)$ are positive). Scheduled transmissions can be blocked as due to some circuit allocation policy. Blocked transmissions are cleared.

The above description fits in the framework of section 2 by setting $\sigma_h = 1/\lambda(m,n)$ and $\tau_h = 1/\mu(m,n)$ for $h=(m,n)$, saying that a connection is idle when no transmission over this connection takes place and active when it is busy, and assuming a circuit allocation policy which restricts the state of busy connections to some "coordinate convex" region G. We give some examples below.

Example 3.1 (Limited total number of circuits) (cf. [4]). For a given state H of busy connections let n_A , n_B and $n_{A,B}$ denote the number of busy connections within A, within B and in between A and B respectively. Assume finite numbers of LA and LB local circuits within A and B and S circuits in between A and B. Then the model of [4] is included by

$$C = \{H \mid n_A \leq LA, n_B \leq LB, n_{A,B} \leq S\}$$

for the dedicated allocation policy with separate circuits for local and long-distance transmissions and by

$$C = \{H \mid n_A \leq LA+S, n_B \leq LB+S, 0 \leq n_{A,B} \leq S-(n_A-LA)^+-(n_B-LB)^+\},$$

where $(y)^+ = 0$ for $y \leq 0$ and $y^+ = y$ for $y > 0$, for the shared allocation policy in

which the inter-MAN circuits are shared among local and long-distance calls. As another shared allocation policy, each long-distance connection may require a local circuit within each local area, which is reflected by

$$C = \{H \mid n_A + n_{A,B} \leq LA, n_B + n_{A,B} \leq LB, n_{A,B} \leq S\}.$$

Example 3.2 (Limited in/output connections). Assume that subscriber m has the constraint that no more than O_m outgoing calls can take place at the same time. Then the examples 3.1 remain valid with the additional restriction to C of:

$$\sum_{\{n: (m,n) \in H\}} \leq O_m \quad (\forall m).$$

Similarly, input constraints, say I_n for subscriber n , are realized by

$$\sum_{\{m: (m,n) \in H\}} \leq I_n \quad (\forall n).$$

Example 3.3 (Excluding connections). Certain connections may have to be excluded to be busy at the same time. For example, exclusion of busy connections (m,n) and (n,m) at the same time reflects one-way communication systems such as in air traffic. The corresponding set of admissible states is "coordinate convex" by:

$$1((m,n) \in H) + 1((n,m) \in H) \leq 1 \quad (\forall (n,m)).$$

References

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