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**FLOWERS OF SPEECH IN ECONOMETRICS?
THE QUEST FOR DHSY**

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Comments welcome.

ABSTRACT

The purpose of the paper is to extend recent results of Muellbauer (1986) for the life cycle consumption model under rational habit formation. For the model with finite time horizon, it is shown that an arbitrary autoregressive integrated moving average (ARIMA) process for consumption is obtained by choosing an appropriate pattern of rational habits. The model with a moving planning horizon is analyzed for a special form of rational habits that leads to a specification in the four period difference operator. It is shown that under some specific assumptions the model will yield the basic mechanism underlying the consumption function of Davidson, Hendry, Srba and Yeo (1978).

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Section 1 Introduction.

Among the many articles that deal with extensions and refinements of Modigliani and Brumberg's (1955) life cycle consumption hypothesis, an important contribution is due to Hall (1978). He formulates the life cycle hypothesis as an intertemporal decision problem and shows that the first order conditions for an intertemporal optimum imply a first order autoregressive process for the marginal utility of consumption. Recently, Muellbauer (1986) has put forward a model that establishes a synthesis between the life cycle hypothesis and habit formation. He investigates two kinds of habit formation, myopic and rational, and gives empirical evidence for the U.S. which favours the presence of myopic habits. For an extensive historical overview of the main relevant contributions both in the cross section and the time series context, I refer to his paper. Winder (1987) extends the results for the case of rational habits. He shows that for a general pattern of habits and an exponential utility function, an arbitrary autoregressive integrated moving average (ARIMA) scheme for consumption may be obtained. As a special case it is shown what structure should be imposed on the preferences to arrive at a model in annual differences. The empirical analysis carried out with data on quarterly seasonally unadjusted nondurable consumption for the Netherlands, suggests the presence of rational habits. In line with Hall's approach, many authors have tested the life cycle model by examining the predictive power of the information set assumed to be used by the consumer. The results for the life cycle model extended for the presence of habits illustrate that checking the significance of past realizations of consumption is not so much a test of the life cycle model as a test of rational habit formation.

Both Muellbauer (1986) and Winder (1987) assume that the consumer uses in the utility maximization problem a infinite life time. This paper considers two alternative assumptions concerning the planning horizon, namely a finite one and a moving one. An obvious reason for postulating a finite life time is the observation of mortality and this paper asks whether the assumption of a finite planning horizon affects the conclusions of Winder (1987). Palm and Winder (1987a) has put forward the model with

moving planning horizon. They argue that it is not unrealistic to imagine that the consumer will neglect periods far ahead in the future on which available information is scarce and unreliable, and will confine himself to more trustworthy information on the near future. They assume that the consumer shifts the planning horizon further ahead in the future as time goes on. The modified model leads to a relationship between income and consumption which is highly similar to the mechanism underlying the consumption function proposed by Davidson, Hendry, Srba and Yeo (1978). More specifically, as a result of adjusting the planning horizon, an error correction term has to be included in the consumption function. As no error is involved from the side of the consumer, they argue that it is more appropriate to speak about a correction term.

Throughout the paper we make the assumption of rational expectations, that is, we assume that the subjective distribution of income used in the utility maximization problem coincides with the actual distribution. The paper is organized as follows. In section 2 we analyze the life cycle model for the exponential utility function. We develop a procedure to find the closed form solution in the case that the preference structure exhibits a general pattern of rational habits. The framework is the same as that of Winder (1987). In this paper it is assumed too, that the consumer uses only information on expected future labour income. This assumption differs from the one often made, when consumers are assumed to take into account the complete distribution of labour income. It will be shown that conditional on the existence of an interior solution, a general pattern of rational habits will lead to an arbitrary ARIMA process for consumption. The only difference with the result derived in Winder (1987) for an infinite time horizon is that the drift parameter and the variance of the consumption process become age/time dependent.

In section 3 we investigate the model with moving planning horizon for a special form of habits that yields a model in the four period difference operator. In the first subsection we derive the univariate stochastic process for the four period change in consumption when the annual change in income is generated by an ARMA(p,q) process. In the structural econometric modelling and time series analysis (SEMTSA) approach to econometric modelling put forward by Zellner and Palm (1974), checking the implications of the economic theory for the univariate stochastic processes is an

essential feature. The analysis may provide the skeleton of simple ARIMA schemes for consumption with economic flesh, and may shed some light on the frequently encountered similarities of the stochastic processes for income and consumption (see e.g. Prothero and Wallis (1976)). Moreover, it is shown that the drift parameter of the consumption process is proportional to that of the income process. Hence, an unanticipated change in the latter has well defined implications for the former. In the second subsection we show that when the annual change of income follows an autoregressive process of order 1, the model leads to a relationship between consumption and income that is similar to the mechanism underlying the consumption function of Davidson, Hendry, Srba and Yeo (1978). More specifically, in each quarter of a year the consumer spends the same as he spent in that quarter of the previous year, modified by a proportion of the annual change in income and by whether that change is itself increasing or decreasing, and by the error correction term. Finally, section 4 concludes the study.

Section 2 The model with finite planning horizon.

In this section we discuss the life cycle consumption model. At each time period t , the consumer is assumed to maximize his life time utility subject to the life time budget constraint

$$\begin{aligned} \text{MAX } & \bar{u}(c_t, c_{t+1}, \dots, c_T) \\ \text{S.T. } & \sum_{i=0}^{T-t} (1+r)^{-i} c_{t+i} \leq (1+r)a_{t-1} + \sum_{i=0}^{T-t} (1+r)^{-i} E(y_{t+i} | I_t) \\ & c_{t+i} \geq 0, \quad i=0, \dots, T-t, \end{aligned} \quad (2.1)$$

where c_t and y_t are real consumption and real labour income respectively, a_{t-1} real financial wealth. T denotes the life time, and r is the real interest rate which is assumed to be constant ($0 < r < 1$). E stands for the expectations operator and I_t is the set of information available at time t . The only source of uncertainty concerns future labour income and it is assumed that the consumer knows the value of y_t when making his decision c_t . Hence $E(y_t | I_t) = y_t$. Notice that $\bar{u}$ may depend on additional predetermined variables. Examples are for instance taste shifters or, in case of habit formation, past realizations of consumption.

When $\bar{u}$ is quasi-concave and the consumer is never satiated in at least one c_{t+i} , the quasi saddle point (QSP) characterization for an optimum yields a necessary and sufficient condition for a global maximum (see e.g. Takayama (1985), p.135). When we restrict ourselves to an interior solution the QSP characterization reduces to the familiar condition

$$\frac{\partial \bar{u}}{\partial c} \bigg|_{\hat{c}} = \hat{\lambda} p', \quad (2.2a)$$

$$\begin{aligned} p' \hat{c} &= (1+r)a_{t-1} + \sum_{i=0}^{T-t} E(y_{t+i} | I_t) \\ \hat{\lambda} &> 0, \quad \hat{c} > 0, \end{aligned} \quad (2.2b)$$

where λ is the multiplier associated with the life time budget constraint, c and p denote the $(T-t+1)$ -vectors $(c_t, c_{t+1}, \dots, c_T)'$ and $(1, (1+r)^{-1}, \dots, (1+r)^{-(T-t)})'$ respectively and $(\hat{c}, \hat{\lambda})$ represents the solution of (2.2). Moreover, when $\bar{u}$ is strictly quasi-concave $\hat{c}$ is a unique global maximum.

To arrive at concrete results, it becomes necessary to impose some

structure on the preferences. In this paper we investigate

$$\bar{u}(c_t, \dots, c_T) = \sum_{i=0}^{T-t} \beta^i U(\Phi(L)c_{t+i}) \quad , \quad (2.3)$$

with $U' > 0$ and $U'' < 0$, where U' and U'' are the first and second derivatives of U respectively, β is the preference parameter, $0 < \beta < 1$, and $\Phi(L)$ is a polynomial of order p in the lag operator L

$$\Phi(L) = 1 - \varphi_1 L - \dots - \varphi_p L^p .$$

Later we discuss the generalization of (2.3) to the case that the preference structure depends on an infinite number of past realizations of consumption. Winder (1987) studies the utility function (2.3) with infinite planning horizon, that is $T = \infty$. Notice that

$$\frac{\partial \bar{u}}{\partial c_T} = \beta^{T-t} U'(\Phi(L)c_T) > 0 ,$$

so that the consumer is never satiated in at least one commodity. It can easily be seen that condition (2.2a) implies

$$\frac{\partial \bar{u}}{\partial c_{t+i}} = (1+r)^{T-i} \frac{\partial \bar{u}}{\partial c_{t+T}} \quad , \quad i=0,1,\dots,T-t-1 .$$

Hence, the necessary condition for the existence of an interior solution, that is

$$\left. \frac{\partial \bar{u}}{\partial c_{t+i}} \right|_c > 0 \quad , \quad i=0,1,\dots,T-t ,$$

is satisfied. This condition is of course not sufficient to guarantee that corner solutions are excluded. Derivation of the restrictions resulting from the "interior solution postulate", on the parameters of the utility function (2.3) is however far from being straightforward. Hence, we implicitly assume that the parameters φ_i are such that the existence of an interior solution is not excluded. Expression (2.3) shows that the decision c_t is affected by past choices of consumption. Past consumption is assumed to influence current consumption in a way that corresponds to

the rational habits formulation.

For the subsequent analysis it proves convenient to define

$$c_{t+i}^* = \Phi(L)c_{t+i}^*$$

and

$$c^* = (c_t^*, c_{t+1}^*, \dots, c_T^*)$$

When we define

$$\Psi(L) = \Phi(L)^{-1} = \sum_{i=0}^{\infty} \psi_i L^i, \quad \psi_0 = 1 \quad (2.4)$$

we have

$$c_{t+i} = \Psi(L)c_{t+i}^* = \sum_{j=0}^i \psi_{i-j} c_{t+j}^* + \sum_{j=i+1}^{\infty} \psi_j c_{t+i-j}^*$$

or in matrix notation

$$c = Ac^* + \bar{c},$$

where A is the upper triangular matrix with $A_{ij} = \psi_{i-j}$, $i \geq j$ and $\bar{c}$ is the $(T-t+1)$ -vector $(\bar{c}_t, \bar{c}_{t+1}, \dots, \bar{c}_T)'$ with

$$\bar{c}_{t+i} = \sum_{j=i+1}^{\infty} \psi_j c_{t-j+i}^*.$$

Notice that $\bar{c}$ depends only on past realizations of consumption.

When we define

$$\tilde{u}^*(c^*) = \tilde{u}(Ac^* + \bar{c}) = \sum_{i=0}^{T-t} \beta^i U(c_{t+i}^*)$$

we have

$$\frac{\partial \tilde{u}}{\partial c} = \frac{\partial \tilde{u}^*}{\partial c^*} \frac{\partial c^*}{\partial c} = \frac{\partial \tilde{u}^*}{\partial c^*} A^{-1}$$

and condition (2.2) passes into

$$\frac{\partial \tilde{u}^*}{\partial c^*} \Big|_{\hat{c}^* = A^{-1}(\hat{c} - \bar{c})} = \hat{\lambda} p' A \quad (2.5a)$$

$$p' A \hat{c}^* = (1+r)a_{t-1} + \sum_{i=0}^{T-t} (1+r)^{-i} E\{y_{t+i} | I_t\} - p' \bar{c} \quad (2.5b)$$

Moreover, because

$$\frac{\partial^2 \tilde{u}}{\partial c \partial c'} = A^{-1} \frac{\partial^2 \tilde{u}^*}{\partial c^* \partial c^*} A^{-1} \quad \text{and} \quad \frac{\partial^2 \tilde{u}^*}{\partial c^* \partial c^*} \quad \text{is}$$

negative definite, we conclude that $u(c)$ is strictly concave and hence strictly quasi-concave.

We proceed by examining (2.5), which yields $T-t+2$ equations to solve the $T-t+2$ unknown variables $\hat{c}_t, \dots, \hat{c}_T, \hat{\lambda}$. The $(i+1)^{th}$ equation of (2.5a) reads as

$$\beta^i U'(c_{t+i}^*) = \lambda (1+r)^{-i} \eta_{T-t-i}, \quad (2.6)$$

where

$$\eta_k = \sum_{j=0}^k (1+r)^{-j} \psi_j.$$

Using expression (2.6) for $i=0$, we find

$$\beta^i U'(c_{t+i}^*) = (1+r)^{-i} \eta_{T-t-i} \eta_{T-t}^{-1} U'(c_t^*), \quad i=1, 2, \dots, T-t. \quad (2.7)$$

To arrive at an operational model, it is necessary to make a specific choice for U . In this paper we investigate

$$U(c) = -\gamma^{-1} \exp(-\gamma c) \quad , \quad \gamma > 0. \quad (2.8)$$

Expression (2.7) can be rewritten for utility function (2.8) as

$$c_{t+i}^* = c_t^* + i\gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-i} \eta_{T-t}^{-1}] \quad , \quad i=1, \dots, T-t. \quad (2.9)$$

After substitution of (2.9) into the life time budget constraint (2.5b), we get for the decision c_t^*

$$\begin{aligned}
 c_t^* &= \sum_{k=0}^{T-t} \eta_{T-t-k} (1+r)^{-k} + \sum_{k=1}^{T-t} \eta_{T-t-k} (1+r)^{-k} (k\gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-k} \eta_{T-t}^{-1}]) \\
 &= (1+r)a_{t-1} + \sum_{i=0}^{T-t} (1+r)^{-i} E(y_{t+i} | I_t) - \sum_{k=0}^{T-t} (1+r)^{-k} \sum_{j=k+1}^{\infty} \psi_j c_{t+k-j}^* \quad (2.10)
 \end{aligned}$$

Substituting $c_t^* = \Phi(L)c_t$ into formula (2.10) yields the current consumption decision c_t as a function of income, future income expectations, wealth and past consumption. In line with Brown (1952), the latter may be interpreted as the influence of habits on consumer behaviour.

It is implicitly assumed that the consumption decision c_t determined by expression (2.10), is positive. The feasibility depends among other things on the specific values of life time wealth and past consumption levels. Notice also that it is required that the implied planned consumption levels c_{t+i} determined by (2.9), correspond to an interior solution. Expression (2.9) reveals in this respect that we need to impose the necessary condition

$$\eta_{T-t-i} \eta_T^{-1} > 0, \quad i=1, \dots, T-t. \quad (2.11)$$

This leads to restrictions on the ψ -coefficients of (2.4) and hence on the φ -coefficients of the utility function (2.3). We have not been successful in deriving the form of the restrictions in the model with a general pattern of rational habits. For the specific form of rational habits discussed in the next section, the restrictions (2.11) are satisfied. To investigate the dynamics in consumption, it is convenient to relate c_t to c_{t+1} . Carrying out the same operations as before for the model for the next period leads to

$$\begin{aligned}
 c_{t+1}^* &= \sum_{k=0}^{T-t-1} (1+r)^{-k} \eta_{T-t-1-k} + \\
 &+ \sum_{k=1}^{T-t-1} (1+r)^{-k} \eta_{T-t-1-k} (k\gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-1-k} \eta_{T-t-1}^{-1}]) =
 \end{aligned}$$

$$(1+r)a_t + \sum_{i=0}^{T-t-1} (1+r)^{-i} E(y_{t+1+i} | I_{t+1}) - \sum_{k=0}^{T-t-1} (1+r)^{-k} \sum_{j=k+1}^{\infty} \psi_j c_{t+1+k-j}^* \quad (2.12)$$

Dividing (2.12) by $(1+r)$, substituting $a_t = (1+r)a_{t-1} + y_t - c_t$, subtracting (2.10) and using

$$-c_t = (1+r)^{-1} \left(\sum_{k=0}^{T-t-1} (1+r)^{-k} \sum_{j=k+1}^{\infty} \psi_j c_{t+1+k-j}^* \right) +$$

$$\sum_{k=0}^{T-t} (1+r)^{-k} \sum_{j=k+1}^{\infty} \psi_j c_{t+k-j}^* = -c_t^* \eta_{T-t}$$

and

$$(1+r)^{-1} \left(\sum_{k=1}^{T-t-1} (1+r)^{-k} \eta_{T-t-1-k} (k\gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-1-k} \eta_{T-t-1}^{-1}]) \right)$$

$$- \sum_{k=1}^{T-t} (1+r)^{-k} \eta_{T-t-k} (k\gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-k} \eta_{T-t}^{-1}]) =$$

$$= - \sum_{k=1}^{T-t} (1+r)^{-k} \eta_{T-t-k} (\gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-1} \eta_{T-t}^{-1}])$$

leads to

$$c_{t+1}^* - c_t^* = \gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-1} \eta_{T-t}^{-1}] +$$

$$\left(\sum_{k=0}^{T-t-1} (1+r)^{-k} \eta_{T-t-1-k} \right)^{-1} \left(\sum_{i=0}^{T-t-1} (1+r)^{-i} (E(y_{t+1+i} | I_{t+1}) - E(y_{t+1+i} | I_t)) \right) \quad (2.13)$$

When we substitute next

$$c_{t+1}^* - c_t^* = \Phi(L) \Delta c_{t+1}$$

into (2.13) and define the consumption innovation ε_{t+1} as $\varepsilon_{t+1} = c_{t+1} -$

$E(c_{t+1} | I_t)$, we have

$$\Phi(L)\Delta c_{t+1} = \gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-t-1} \eta_{T-t}^{-1}] + \varepsilon_{t+1} \quad (2.14)$$

with

$$\varepsilon_{t+1} = \left(\sum_{k=0}^{T-t-1} (1+r)^{-k} \eta_{T-t-1-k} \right)^{-1} \left(\sum_{i=0}^{T-t-1} (1+r)^{-i} (E(y_{t+1+i} | I_{t+1}) - E(y_{t+1+i} | I_t)) \right). \quad (2.15)$$

Consumption follows an autoregressive integrated (ARI) process. When unit roots are not incompatible with the assumption that an interior solution exists, the order of integration may be larger than 1 for an appropriate choice of the lag polynomial $\Phi(L)$ (only integration of order zero is excluded). Notice that a model in the s-period difference operator Δ_s is obtained when $\Phi(L) = 1 + L + \dots + L^{s-1}$. The lag polynomial $\Phi(L) = 1 + L + L^2 + L^3$ yields for instance a consumption model in annual differences. Obviously, this polynomial can play an important role in modeling seasonally unadjusted consumption series. In the next section the model with moving planning horizon will be examined for this particular choice of $\Phi(L)$.

Expressions (2.14) and (2.15) reveal that both the drift parameter of the stochastic process and the variance of the consumption innovation are age/time dependent. Estimation of model (2.14) with a constant variance and a constant drift is expected to be appropriate when the age structure of the population and the income distribution over different age groups are fairly stable over time.

Palm and Winder (1987a) investigate the model (2.1), (2.3) with utility function (2.8) without habit formation. In that case $\Phi(L) = \eta_k = 1$ for all k and (2.14) and (2.15) specialize to

$$\Delta c_{t+1} = \gamma^{-1} \ln[\beta(1+r)] + \varepsilon_{t+1}$$

with

$$\varepsilon_{t+1} = \left(\sum_{k=0}^{T-t-1} (1+r)^{-k} \right)^{-1} \left(\sum_{i=0}^{T-t-1} (1+r)^{-i} (E(y_{t+1+i} | I_{t+1}) - E(y_{t+1+i} | I_t)) \right).$$

Many observations made in that paper remain valid for the extended model

investigated here. An unanticipated change in the process of the exogenous variable y_t has for instance definite effects on the model for consumption. In line with Lucas (1976) these effects can be traced by using expressions (2.10) and (2.12). The empirical analyses carried out in Palm and Winder (1987a) and Winder (1987) provide illustrations how the implications of the structural changes in the income process can be handled.

In the model discussed above the consumer is assumed to have a preference structure that depends on a finite number of past realizations of consumption. Pollak (1970) mentions the possibility that the utility function includes all past values of consumption and (2.3) may be generalized by replacing the utility function argument $\Phi(L)c_{t+i}$ by $\Phi(L)\Theta(L)^{-1}c_{t+i}$, where $\Theta(L)$ is a finite order lag polynomial. It seems reasonable to impose the additional restriction that the roots of $\Theta(L)=0$ lie outside the unit circle. By this restriction we are assured that the consumer attaches declining weights to the very past of consumption. When we define $\Psi(L)$ in (2.4) as

$$\Psi(L) = \Phi(L)^{-1}\Theta(L)$$

it can easily be seen that the foregoing analysis remains valid. The ultimate result (2.14) passes into

$$\Phi(L)\Delta c_{t+1} = \Theta(L)(\gamma^{-1}\ln[\beta(1+r)] - \gamma^{-1}\ln[\eta_{T-t-1}\eta_{T-t}^{-1}]) + \Theta(L)\varepsilon_{t+1} \quad (2.16)$$

where ε_{t+1} is given by (2.15). Hence, consumption will follow an ARIMA process, the parameters of which correspond to the weights attached to past consumption in the utility function. The results of this section correspond with those obtained by Winder (1987) for the infinite horizon model. The only difference consists in the age-dependency of the drift parameter of the stochastic process and the variance of the consumption innovation for the model discussed in this paper. In both models, the drift parameter depends only on parameters that characterize consumer behaviour. A change in the slope of the consumption line can only be explained in this framework by a change in the parameters of the decision problem determined by (2.1) and (2.3). In Palm and Winder (1987a) it is shown that in the model with moving planning horizon, the drift parameter of the consumption

process is proportional to that of the income process. Hence, an unanticipated change in the slope of the income line will imply a change of the constant term of the consumption model. The next section asks whether this implication remains valid for the extended model with habit formation.

Section 3 The model with moving planning horizon of constant length.

In Palm and Winder (1987) the model with moving planning horizon is introduced as an alternative for the life cycle model. The modified model is capable to relate a change in the slope of the consumption line to a change in the drift parameter of the income process. An attractive feature of the model is that it leads to a consumption function with an error correction term. This term was found to yield favourable empirical results in Davidson, Hendry, Srba and Yeo (1978), where it was derived along completely different lines of reasoning. In this section, we ask whether the assumption of a moving planning horizon together with habit formation will produce a consumption function with error correction term in annual differences. In the light of the analysis carried out in section 2, the special form of rational habits that leads to a model in the four period difference operator Δ_4 is of particular interest. In the first subsection we describe the model and derive the consumption function for an arbitrary income process. Next, we examine the implied univariate stochastic process for consumption when the annual change in income follows an ARMA(p,q) process. In the second subsection we show that when the annual change in income is generated by an AR(1) process, the model leads to a relationship between consumption and income that is highly similar to that of Davidson et. al. (1978).

Section 3.1 The model and the implied univariate stochastic process for consumption.

We assume that the consumer solves at each time period t the utility maximization problem

$$\text{MAX} \quad \sum_{i=0}^T \beta^i U(\Phi(L)c_{t+i}) \quad (3.1)$$

$$\text{S.T.} \quad \sum_{i=0}^T (1+r)^{-i} c_{t+i} = (1+r)a_{t-1} + \sum_{i=0}^T (1+r)^{-i} E(y_{t+i} | I_t)$$

with

$$\Phi(L) = 1 + L + L^2 + L^3$$

The difference with the model (2.1) and (2.3) is that now the length of the planning horizon T is time-independent and shifts as time goes on.

It can easily be checked that

$$\Phi(L)^{-1} = 1 - L + L^4 - L^5 + L^8 - L^9 + \dots$$

and accordingly we have for the ψ -coefficients of (2.4)

$$\begin{aligned} \psi_{4i} &= 1 \\ \psi_{4i+1} &= -1 \\ \psi_{4i+2} &= \psi_{4i+3} = 0, \quad i = 0, 1, 2, \dots \end{aligned} \quad (3.2)$$

Along the lines of section 2, it can be shown that for the utility function (2.8), we have for the consumption decision

$$\begin{aligned} c_t^* \sum_{i=0}^T \eta_{T-i} (1+r)^{-i} + \alpha &= (1+r)a_{t-1} + \sum_{i=0}^T (1+r)^{-i} E(y_{t+i} | I_t) \\ &- \sum_{i=0}^T (1+r)^{-i} \sum_{j=i+1}^{\infty} \psi_j c_{t+i-j}^* \end{aligned} \quad (3.3)$$

where

$$\alpha = \sum_{i=1}^T (1+r)^{-i} \eta_{T-i} (i\gamma^{-1} \ln[\beta(1+r)] - \gamma^{-1} \ln[\eta_{T-1} \eta_T^{-1}])$$

and

$$c_{t+i}^* = \Phi(L)c_{t+i} \quad \text{for all } i.$$

Along the lines of the analysis of section 2 we assume that the consumption decision c_t determined by (3.3) is feasible. As a consequence of the pattern of the ψ -coefficients, it becomes necessary to make an additional

assumption on the length of the planning horizon. Here we make the additional assumption that the consumer uses a time horizon of i years, that is $T=4i$.

After some simple manipulations with the lag polynomial $\Phi(L)$, we find

$$\sum_{j=i+1}^{\infty} \psi_j c_{t+i-j}^* = \begin{array}{ll} -(c_{t-1} + c_{t-2} + c_{t-3}) & , i=0, 4, 8, \dots, T \\ c_{t-3} & , i=1, 5, 9, \dots, T-3 \\ c_{t-2} & , i=2, 6, 10, \dots, T-2 \\ c_{t-1} & , i=3, 7, 11, \dots, T-1 \end{array}$$

and hence after some rearranging

$$\begin{aligned} \sum_{i=0}^T (1+r)^{-i} \sum_{j=i+1}^{\infty} \psi_j c_{t+i-j}^* &= -(1+r)^{-T} (c_{t-1} + c_{t-2} + c_{t-3}) + \\ &+ \tau(T-4) \left\{ \left(-1 + \frac{1}{(1+r)^3} \right) c_{t-1} + \left(-1 + \frac{1}{(1+r)^2} \right) c_{t-2} + \left(-1 + \frac{1}{1+r} \right) c_{t-3} \right\} , \end{aligned} \quad (3.4)$$

where

$$\tau(4k) = \sum_{i=0}^k (1+r)^{-4i} .$$

Also, it follows for the ψ -coefficients given by (3.2) that

$$\sum_{i=0}^T (1+r)^{-i} \eta_{T-i} = \tau(T) . \quad (3.5)$$

Substitution of (3.4) and (3.5) into (3.3) yields for the decision c_t^*

$$\begin{aligned} \tau(T) c_t^* + \alpha = (1+r) a_{t-1} + \sum_{i=0}^T (1+r)^{-i} E(y_{t+i} | I_t) + (1+r)^{-T} (c_{t-1} + c_{t-2} + c_{t-3}) + \\ + \tau(T-4) \left\{ \left(-1 + \frac{1}{(1+r)^3} \right) c_{t-1} + \left(-1 + \frac{1}{(1+r)^2} \right) c_{t-2} + \left(-1 + \frac{1}{1+r} \right) c_{t-3} \right\} . \end{aligned} \quad (3.6)$$

For the next period, the expression for the decision c_{t+1}^* reads like

$$\tau(T) c_{t+1}^* + \alpha = (1+r) a_t + \sum_{i=0}^T (1+r)^{-i} E(y_{t+1+i} | I_{t+1}) + (1+r)^{-T} (c_t + c_{t-1} + c_{t-2}) +$$

$$+ r(T-4) \left\{ \left(1 - \frac{1}{(1+r)^3}\right) c_t + \left(1 - \frac{1}{(1+r)^2}\right) c_{t-1} + \left(1 - \frac{1}{1+r}\right) c_{t-2} \right\} . \quad (3.7)$$

Dividing (3.7) by $1+r$, substituting $a_t = (1+r)a_{t-1} + y_t - c_t$ and subtracting (3.6) leads to

$$\begin{aligned} (c_{t+1}^* - c_t^*) r(T) (1+r)^{-1} &= \left(1 - \frac{1}{1+r}\right) \alpha + \sum_{i=1}^{T+1} (1+r)^{-i} \{E(y_{t+i} | I_{t+1}) - E(y_{t+i} | I_t)\} + \\ &+ (1+r)^{-(T+1)} \{E(y_{t+T+1} | I_t) - c_{t-3}\} , \end{aligned} \quad (3.8)$$

where we have used

$$\begin{aligned} -c_t + r(T-4)(1+r)^{-1} \left\{ \left(1 - \frac{1}{(1+r)^3}\right) c_t + \left(1 - \frac{1}{(1+r)^2}\right) c_{t-1} + \left(1 - \frac{1}{1+r}\right) c_{t-2} \right\} + \\ + (1+r)^{-(T+1)} (c_t + c_{t-1} + c_{t-2}) - (1+r)^{-T} (c_{t-1} + c_{t-2} + c_{t-3}) - \\ - r(T-4) \left\{ \left(1 - \frac{1}{(1+r)^3}\right) c_{t-1} + \left(1 - \frac{1}{(1+r)^2}\right) c_{t-2} + \left(1 - \frac{1}{1+r}\right) c_{t-3} \right\} = \\ = -(\tau(T) - (1+r)^{-1} \tau(T)) c_t^* - (1+r)^{-(T+1)} c_{t-3} . \end{aligned}$$

Substitution of $\Delta c_{t+1}^* = \Delta_4 c_{t+1}$ into (3.8) and multiplying with $(1+r)/\tau(T)$ yields finally

$$\begin{aligned} \Delta_4 c_{t+1} &= \tau(T)^{-1} r \alpha + \tau(T)^{-1} \sum_{i=0}^T (1+r)^{-i} \{E(y_{t+1+i} | I_{t+1}) - E(y_{t+1+i} | I_t)\} + \\ &+ \tau(T)^{-1} (1+r)^{-T} \{E(y_{t+T+1} | I_T) - c_{t-3}\} . \end{aligned} \quad (3.9)$$

An interesting feature of the consumption function (3.9) is the appearance of the error correction term $E(y_{t+T+1}|I_t) - c_{t-3}$. Its presence arises from the adjustment of the planning horizon as time goes on. The introduction of a moving planning horizon provides an alternative explanation for the inclusion of an error correction mechanism in the consumption function. To complete the model for consumption, we have to specify the process for income. In the next subsection we show that when the annual change of income follows an AR(1) process, (3.9) passes into the consumption function of Davidson et. al. (1978). In this section our concern will be to derive the implications of the model (3.1) for the univariate stochastic process of c_t .

The corresponding expression for the consumption decision in period $t-3$ reads like

$$\begin{aligned} \Delta_4 c_{t-3} = & \tau(T)^{-1} r\alpha + \tau(T)^{-1} \sum_{i=0}^T (1+r)^{-i} (E(y_{t-3+i}|I_{t-3}) - E(y_{t-3+i}|I_{t-4})) + \\ & + \tau(T)^{-1} (1+r)^{-T} (E(y_{t+T-3}|I_{t-4}) - c_{t-7}) . \end{aligned} \quad (3.10)$$

Subtracting (3.10) from (3.9) yields

$$\begin{aligned} \Delta_4 c_{t+1} - [1 - (1+r)^{-T} \tau(T)^{-1}] \Delta_4 c_{t-3} = & \tau(T)^{-1} \sum_{i=0}^T (1+r)^{-i} (E(y_{t+i+1}|I_{t+1}) - E(y_{t+i+1}|I_t)) - \\ & \tau(T)^{-1} \sum_{i=0}^T (1+r)^{-i} (E(y_{t-3+i}|I_{t-3}) - E(y_{t-3+i}|I_{t-4})) + \\ & \tau(T)^{-1} (1+r)^{-T} (E(y_{t+T+1}|I_t) - E(y_{t+T-3}|I_{t-4})) . \end{aligned} \quad (3.11)$$

Let us assume that the annual change in income is generated by a stationary process with moving average representation

$$\Delta_4 y_{t+1} = \delta + \sum_{i=0}^{\infty} \pi_i \nu_{t+1-i}, \quad \pi_0=1, \quad \sum_{i=0}^{\infty} \pi_i^2 < \infty, \quad \sigma^2(\nu_{t+1}) = \sigma_\nu^2. \quad (3.12)$$

It can easily be verified from (3.12) that the relevant conditional expectations in (3.11) satisfy

$$\begin{aligned} E(y_{t+1+4i+j} | I_{t+1}) - E(y_{t+1+4i+j} | I_t) &= \sum_{k=0}^i \pi_{4k+j} \nu_{t+1} \\ E(y_{t-3+4i+j} | I_{t-3}) - E(y_{t-3+4i+j} | I_{t-4}) &= \sum_{k=0}^i \pi_{4k+j} \nu_{t-3} \end{aligned} \quad (3.13)$$

with $j=0,1,2,3$ and $i=0,1,2,\dots$, and

$$E(y_{t+T+1} | I_t) - E(y_{t+T-3} | I_{t-4}) = \sum_{j=1}^4 \sum_{k=0}^{T/4} \pi_{4k+j} \nu_{t+1-j} + \pi_0 \nu_{t-3} + \sum_{i=T+5}^{\infty} \pi_i \nu_{t+T+1-i} + \delta.$$

Substitution of (3.13) into (3.11) yields after some rearranging

$$\begin{aligned} \Delta_4 c_{t+1} &= [1 - (1+r)^{-T} \tau(T)^{-1}] \Delta_4 c_{t-3} = \tau(T)^{-1} (1+r)^{-T} \delta + \\ &\tau(T)^{-1} \left[\sum_{i=0}^{(T-4)/4} \sum_{j=0}^3 (1+r)^{-(4i+j)} \sum_{k=0}^i \pi_{4k+j} + (1+r)^{-T} \sum_{k=0}^{T/4} \pi_{4k} \right] \nu_{t+1} + \\ &\tau(T)^{-1} (1+r)^{-T} \sum_{j=1}^3 \sum_{k=0}^{T/4} \pi_{4k+j} \nu_{t+1-j} - \\ &\tau(T)^{-1} \left[\sum_{i=0}^{(T-4)/4} \sum_{j=0}^3 (1+r)^{-(4i+j)} \sum_{k=0}^i \pi_{4k+j} \right] \nu_{t-3} + \\ &\tau(T)^{-1} (1+r)^{-T} \sum_{i=4}^{\infty} \pi_{T+1+i} \nu_{t-i}. \end{aligned} \quad (3.14)$$

From (3.14) it can easily be seen that when the annual change of income is generated by a MA process of order q , $\Delta_4 c_t$ follows an ARMA(4, max(4, q-T)) process. Notice that the ARMA process for $\Delta_4 c_t$ in (3.14) is subject to exclusion restrictions. This can easily be seen for the AR part which

equals $1 - [1 - (1+r)^{-T} \tau(T)^{-1}] L^4$. Conditional on the specific form of the income process, one or more MA parameters may be equal to zero. It can easily be verified that when $\Delta_4 y_t$ satisfies

$$\Delta_4 y_{t+1} = \delta + \sum_{i=0}^k \pi_{4i} \nu_{t+1-4i}$$

with $k \in (0, 1, 2, \dots, (T+4)/4)$, formula (3.14) passes into

$$\Delta_4 c_{t+1} - [1 - (1+r)^{-T} \tau(T)^{-1}] \Delta_4 c_{t-3} = \tau(T)^{-1} (1+r)^{-T} \delta +$$

$$\tau(T)^{-1} \left\{ \sum_{i=0}^{(T-4)/4} (1+r)^{-4i} \sum_{k=0}^i \pi_{4k} + (1+r)^{-T} \sum_{k=0}^{T/4} \pi_{4k} \right\} \nu_{t+1} -$$

$$\tau(T)^{-1} \left\{ \sum_{i=0}^{(T-4)/4} (1+r)^{-4i} \sum_{k=0}^i \pi_{4k} \right\} \nu_{t-3} \quad (3.15)$$

Hence, $\Delta_4 c_t$ follows an ARMA(4,4) process with only the coefficients of the highest lags of the AR and MA parts unequal to zero. To derive the stochastic process for $\Delta_4 c_t$ when $\Delta_4 y_t$ is generated by an ARMA(p,q) model, it becomes necessary to explore the restrictions on the π 's implied by the p+q ARMA parameters. In appendix A it is shown that in that case $\Delta_4 c_t$ follows an ARMA(p+4, max(p+4, max(p-1, q)-T)) process, where the autoregressive part is proportional to that of income.

Models like (3.15) are frequently encountered in time series studies and the analysis of this section may shed some light on the economic story behind the ARMA specifications. Obviously, the theoretical framework can provide useful insight in the structures of the income and consumption process, which may be of some use in the identification stage of the modeling procedure. Moreover, Lucas (1976) has convincingly argued that a theoretical framework enables one to incorporate in his predictions the perturbations in the specification of the consumption process caused by a specific policy. Expression (3.14) shows in this respect that a change in the income line will lead to a change in the constant term of the consumption model. All these results are very similar to those of Palm and Winder (1987a). The empirical analysis carried out in that paper may serve

as an illustration. In Winder (1987) the life cycle model extended for the presence of habits is examined using data on quarterly seasonally unadjusted nondurable consumption per capita for the Netherlands. The chosen specification is an AR(1) process for $\Delta_4 c_t$. As the fall in consumption since 1979 cannot be explained by the model, a structural change in the time preference parameter β is postulated. Notice that the model (3.1) is not capable to obviate this assumption. Expression (3.14) shows that the highest lag of the autoregressive part of the stochastic process is at least 4.

Section 3.2. The Davidson, Hendry, Srba and Yeo model.

In this subsection we will show that if the annual change in income is generated by an autoregressive process of order 1, the consumption function (3.9) is highly similar to the mechanism put forward by Davidson et.al. (1978). More specifically, we assume that

$$\Delta_4 y_{t+1} = \varphi \Delta_4 y_t + \nu_{t+1}$$

For the sake of simplicity we omit the constant term, which does not change the conclusions. It is straightforward to calculate the relevant conditional expectations, which read as

$$E(y_{t+4i+j} | I_{t+1}) - E(y_{t+4i+j} | I_t) = \varphi^{j-1} \sum_{k=0}^i \varphi^{4k} (\Delta_4 y_{t+1} - \varphi \Delta_4 y_t) \quad \begin{matrix} j=1,2,3,4 \\ i=0,1,2,\dots \end{matrix}$$

$$\text{and} \tag{3.16}$$

$$E(y_{t+T+1} | I_t) = y_{t-3} + \varphi \sum_{k=0}^{T/4} \varphi^{4k} \Delta_4 y_t$$

Substitution of (3.16) into (3.9) leads after some rearranging to

$$\Delta_4 c_{t+1} = \alpha_0 + (\alpha_1 - \alpha_2) \Delta_4 y_{t+1} + \alpha_2 \Delta_4 y_{t+1} + \alpha_3 (y_{t-3} - c_{t-3}) \tag{3.17}$$

with

$$\alpha_0 = r(T)^{-1} r \alpha$$

$$\alpha_1 = r(T)^{-1} \left(\sum_{i=0}^{(T-4)/4} \sum_{j=1}^4 (1+r)^{-(4i+j-1)} \varphi^{j-1} \sum_{k=0}^i \varphi^{4k} + (1+r)^{-T} \sum_{k=0}^{T/4} \varphi^{4k} \right)$$

$$\alpha_2 = r(T)^{-1} \varphi \left(\alpha_1 - (1+r)^{-T} \sum_{k=0}^{T/4} \varphi^{4k} \right)$$

$$\alpha_3 = r(T)^{-1} (1+r)^{-T}$$

Expression (3.17) shows that we have the same mechanism as found in Davidson et. al. (1978). With our theoretical model we can determine the sign and size of the coefficients. For α_3 we find that it should be positive and smaller than 1. The sign and size of both $(\alpha_1 - \alpha_2)$ and α_2 depend on the sign of φ . It is easy to show that if $0 < \varphi < 1$ we have $0 < \alpha_1 - \alpha_2 < 1$ and $0 < \alpha_2$, and when $-1 < \varphi < 0$ we have $\alpha_2 < 0$ and $\alpha_1 - \alpha_2 > 1$. In the consumption function of Davidson et.al. the coefficient for the annual change in income is between 0 and 1, and the coefficient for the change of that change is negative. Clearly, their point estimates do not satisfy the plausibility requirements of our theoretical model.

We have also investigated the model under various assumptions concerning the length of the planning horizon. The ultimate consumption function differs only from (3.17) with respect to the lag of the correction term. It can be shown that for a time horizon T such that there is an i_0 with $T = 4i_0 + k$, $k=0,1,2,3$, the correction term reads like $(y_{t-3+k} - c_{t-3+k})$. The coefficients of (3.17) need slight revision for a choice of k unequal to 0 but leaves the inference with respect to the sign and size of the coefficients intact.

Finally, it should be remarked that there remain important differences between specification (3.17) and the consumption function found by Davidson et. al.. The main differences concern the role of the inflation variable and the log-linear functional form. We have the impression that with respect to the functional form, a possible solution might be the use of the utility function with constant relative risk aversion. Palm and Winder (1987b) have investigated this utility function which leads to a log-linear specification for the consumption model. The paper of Muellbauer (1986) however, shows that an analysis of the model extended for the presence of

habits with this utility function is far from being straightforward.

Section 4 Summary.

In this paper we established a synthesis between the life cycle theory and rational habit formation in consumption. Two options concerning the planning horizon were considered, namely a finite one and a moving one.

For the model with finite planning horizon it was shown that an arbitrary ARIMA process for consumption is obtained by choosing an appropriate pattern of rational habits. In the analysis we implicitly assumed that an interior solution existed. However, as a result of the non-negativity constraints on consumption, the possibility of corner solutions can not be precluded. Exclusion of this possibility may lead to restrictions on the parameters of the utility function and hence on the parameters of the ARIMA process. This may lead to the conclusion that some ARMA processes are incompatible with the model of intertemporal optimization examined in section 2.

In the second part we investigated the model with moving planning horizon for a special pattern of rational habits that yields a model in four period differences. We derived the consumption function for an arbitrary income process. The only requirement is that the first (conditional) moments of the income process exist. As a consequence of adjusting the planning horizon, an error correction term has to be included in the consumption function. In our framework no error is involved from the side of the consumer, and therefore we prefer to label it as a correction term. Next, we derived the implied univariate stochastic process for consumption when the annual change in income is generated by an ARMA(p,q) model and we argued that the results can provide useful insight in the structures of the income and consumption processes, which may be of some use in the identification stage of a time series modelling procedure. Finally, we showed that when the annual change in income follows an AR(1) process, the model with moving planning horizon leads to a relationship between consumption and income which is highly similar to the consumption function obtained by Davidson, Hendry, Srba and Yeo (1978).

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Appendix A The univariate stochastic process for consumption in the model with moving planning horizon.

We consider a stationary invertible ARMA(p,q) process for the annual change in income

$$\Phi(L)\Delta_4 y_t = \Theta(L)\nu_t, \text{ with } E(\nu_t)=0 \text{ and } \sigma^2(\nu_t)=\sigma_\nu^2. \quad (A.1)$$

The lag polynomials $\Phi(L)$ and $\Theta(L)$ are defined as

$$\Phi(L) = \varphi_0 - \varphi_1 L - \dots - \varphi_p L^p, \quad \varphi_0=1$$

and

$$\Theta(L) = \theta_0 - \theta_1 L - \dots - \theta_q L^q, \quad \theta_0=1$$

respectively. The MA(∞) representation is denoted as

$$\Delta_4 y_t = \Pi(L)\nu_t \quad (A.2)$$

with

$$\Pi(L) = \sum_{i=0}^{\infty} \pi_i L^i, \quad \pi_0=1.$$

From (A.1) and (A.2) follows

$$\Phi(L)\Pi(L) = \Theta(L). \quad (A.3)$$

Relationship (A.3) can be used to trace the restrictions on the parameters π_i implied by the p+q ARMA parameters. It is straightforward to show that (A.3) implies

$$\pi_j = \varphi_1 \pi_{j-1} + \dots + \varphi_p \pi_{j-p} \quad \text{for all } j \geq \max(p, q+1). \quad (\text{A.4})$$

The parameters π_j , $j \geq \max(p, q+1)$, are generated by a p^{th} order homogeneous difference equation. When we define μ_t as the MA part of (3.14), we have

$$\mu_t = \sum_{j=0}^{\infty} \alpha_j \nu_{t-j}$$

with

$$\alpha_0 = \tau(T)^{-1} \left[\sum_{i=0}^{(T-4)/4} \sum_{j=0}^3 (1+r)^{-(4i+j)} \sum_{k=0}^i \pi_{4k+j} + (1+r)^{-T} \sum_{k=0}^{T/4} \pi_{4k} \right]$$

$$\alpha_j = \tau(T)^{-1} (1+r)^{-T} \sum_{k=0}^{T/4} \pi_{4k+j}, \quad j=1, 2, 3$$

$$\alpha_4 = -\tau(T)^{-1} \left[\sum_{i=0}^{(T-4)/4} \sum_{j=0}^3 (1+r)^{-(4i+j)} \sum_{k=0}^i \pi_{4k+j} \right]$$

$$\alpha_j = \tau(T)^{-1} (1+r)^{-T} \pi_{T+j}, \quad j \geq 5.$$

Calculating the autocovariance function for μ_t yields for all $i \geq 5$

$$E(\mu_t \mu_{t-i}) = \sigma_\nu^2 \tau(T)^{-1} (1+r)^{-T} \sum_{j=0}^{\infty} \pi_{T+i+j} \alpha_j. \quad (\text{A.5})$$

For every i satisfying $T+i \geq \max(p, q+1)$, we can use (A.4) to rewrite (A.5) as

$$\begin{aligned} E(\mu_t \mu_{t-i}) &= \sigma_\nu^2 \tau(T)^{-1} (1+r)^{-T} \sum_{j=0}^{\infty} \sum_{k=1}^p \varphi_k \pi_{T+i+j-k} \alpha_j \\ &= \sum_{k=1}^p \varphi_k \left[\sigma_\nu^2 \tau(T)^{-1} (1+r)^{-T} \sum_{j=0}^{\infty} \pi_{T+i+j-k} \alpha_j \right]. \end{aligned} \quad (\text{A.6})$$

For every i satisfying in addition $T+i-p \geq T+5$, substituting of (A.4) into (A.6) gives as a result

$$E(\mu_t \mu_{t-i}) = \sum_{k=1}^p \phi_k E(\mu_t \mu_{t-(i-k)}) \quad (A.7)$$

From the requirements in (A.5), (A.6) and (A.7), we see that for all $i \geq \max(p+5, \max(p, q+1)-T)$ the autocovariances of μ_t are generated by a p^{th} order homogeneous difference equation. For an ARMA(r,s) process the autocovariances γ_j are generated by an r^{th} order homogeneous difference equation for all $j \geq s+1$. Because the autocovariance function determines the order of a stationary stochastic process, we conclude from (A.7) that μ_t follows an ARMA(p, $\max(p+4, \max(p-1, q)-T)$) process, where the AR part coincides with the one for the income process. Say

$$\Phi(L)\mu_t = \tilde{\Theta}(L)\zeta_t, \quad E(\zeta_t) = 0 \text{ and } \sigma^2(\zeta_t) = \sigma_\zeta^2,$$

or in MA representation

$$\mu_t = \Phi(L)^{-1} \tilde{\Phi}(L)\zeta_t. \quad (A.8)$$

Substitution of (A.8) in (3.14) leads to the conclusion that $\Delta_4 c_t$ is generated by an ARMA(p+4, $\max(p+4, \max(p-1, q)-T)$) process. Notice that this ARMA process is subject to exclusion restrictions. The AR part factorizes as

$$\Phi(L)(1 - [1 - \tau(T)]^{-1}(1+r)^{-T}]L^4)$$

and the example discussed in section 3 illustrates that conditional on the specific form of the income process (A.1) one or more of the MA parameters may be equal to zero.