

A THEORY OF DISPLACED IDEALS

An Analysis of Interdependent
Decisions via Non-linear
Multi-objective Optimization.

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1. Introduction.

A usual paradigm in traditional economics is the assumption of an independent rational decision-maker who attempts to maximize his welfare (utility, profits etc.), given a set of side-conditions on the state and control space. On the basis of this starting-point the optimal strategies of the decision-maker can be identified. Furthermore, the same approach can be used as an ex post analysis to assess by means of a revealed preference hypothesis the reaction parameters of the decision-maker (see for an application to macro-economic public policy Nijkamp and Somermeijer [1971] and to consumer behaviour Blokland [1976]).

A considerable part of the optimization framework in economics is based on independent behaviour of a decision-maker. Clearly, it should be admitted that, in the past, several theories have been developed in which the economic actions of market participants are assumed to be co-determined by decisions of other actors. A well-known example is Duesenberry's relative income hypothesis, which states that the difference in consumption behaviour among households can be explained by differences in the level of relative income i.e., income in relation to the standard of living of a socio-economic group one is accustomed to (see Duesenberry [1952]). The phenomenon of 'keeping up with the Joneses' is also closely related to the relative income hypothesis.

Another example of interdependent market behaviour is a situation of limited competition (such as a duopoly or an oligopoly), in which the outcome of the market process is determined by mutually interdependent strategies of all participants (like Cournot and Von Stackelberg strategies). The final outcome is the result of a game situation on which basis the equilibrium conditions can be analysed (see also Paelinck and Nijkamp [1976]).

A new contribution to the discussion on interdependent behaviour was recently provided by Leibenstein [1976]. Leibenstein claims that the traditional neo-classical micro-economic theory has several fundamental shortcomings, inter alia by assuming independent decision units (such as families or firms) as the methodological basis for rational, behavioural hypotheses. In his opinion, individuals should be regarded as the basic elements of a decision framework. Instead of a molecular view of these individual units, however, Leibenstein makes a plea for a more psychologically oriented analysis of decision processes. He claims that psychological factors like motivation and effort are extremely important elements in human decision-making, although their influence is hardly quantifiable. Therefore, he calls these elements X-efficiency factors. The implications of Leibenstein's analysis are rather far-

reaching. If instead of households and firms individuals are assumed to be the really motivated units, then it can be demonstrated that utility maximization by households may be consistent with rational behaviour, but this is not a necessary result. Analogously, profit maximization of a firm is not a priori consistent with the motivations of the individuals in a firm.

Consequently, Leibenstein pays much attention to mutual interactions of market participants. He offers a detailed analysis of interdependent actions such as bandwagon effects, snob effects and Veblen effects, in which also game-theoretic elements may play a role (among others in the prisoner's dilemma case). Unfortunately, Leibenstein has not elaborated his analysis in an operational sense.

In our opinion, it is worth while to seek for a more operational analysis of interdependent decision-making. In the present paper, the problem of interdependent choices will be analysed along two different, but complementary lines. The first approach focusses on market participants or actors with mutually conflicting objectives, so that the (optimal) independent decisions of the one actor affect the well-being of the other ones. Some equilibrium strategies for conflicting actions based on 'displaced ideals' will be discussed here. The method of displaced ideals employs the ideal solutions of a choice problem as reference points in order to find a minimum discrepancy with respect to the set of feasible or Pareto solutions of the choice problem at hand. This situation will be illustrated by the well-known dilemma economic growth - environmental preservation (see section 4).

The second and complementary approach is oriented to a spatial variant of the 'keeping up with the Joneses' phenomenon. In this situation certain prespecified achievement levels of goal variables (the 'ideal values') of the one actor (region) are co-determined by the optimal (and observed) levels of the same variables of the other actors. By repeating this situation during several periods, the concept of 'displaced ideals' may again be introduced. This 'keeping up with the Joneses' effect will be illustrated by means of a simple two-region economic policy model which is an extension of the preceding economic-environmental model (see section 5).

The formal mathematical framework to describe interdependent decision-making is based on multi-objective optimization, because both a situation of mutually conflicting objectives (in case of diverging strategies of actors), and of mutually complementary objectives (in case of adaptive strategies of actors), can be formally described as a decision problem with multiple objectives. Therefore, section 2 of this paper will give a brief introduction

to multi-objective optimization. Special attention will be paid to the possibility to solve multi-objective programs by means of interactive strategies. A powerful interactive tool in multi-objective decision-making is the method of 'displaced ideals' referred to above.

In the present paper, non-linear objective functions and non-linear discrepancy measures will be introduced. Their specifications are based on power functions and give rise to non-linear programming models. Therefore, in section 3 a brief introduction will be given to a programming method which is extremely appropriate for this type of non-linearities, viz. geometric programming. This method appears to provide also a new framework for solving generalized non-linear multi-objective models.

After the presentation of the necessary tools in section 2 and 3, in section 4 and 5 some simplified regional environmental models will be presented in which both a situation of conflicting priorities (growth versus pollution) and of mutually adaptive priorities (a situation of successive displaced ideals) will be analysed.

2. Multi-objective Optimization.

The traditional measure for the economic health of a country or region is average income. Due to the wide variety of external effects related to the post-war growth, during the last decade the insight has grown that welfare is essentially a multi-dimensional concept including inter alia income, growth, environmental quality, distributional equity, supply of public facilities and so forth. In other words, the welfare of a country or region should be represented by a vector profile instead of by a scalar (see for a further discussion Hafkamp and Nijkamp [1978] and Klaassen[1978]).

In the context of decision models, a multi-dimensional view of welfare leads to a plea for multi-objective optimization models, in which multiple (conflicting) objective functions are to be optimized simultaneously. The reasons for the existence of multiple objective functions may be: the presence of non-commensurable objectives, the presence of different interest groupings or the presence of spill-over effects.

In general formal terms, a multi-objective optimization model may be represented as:

$$(2.1.) \quad \begin{cases} \max \quad \underline{w}(\underline{x}) \\ \underline{x} \in K \end{cases}$$

where $\underline{w}(\underline{x})$ is a vector of objective functions, $\underline{x}$ a vector of decision variables and K a feasible area.

There is a large set of methods to analyse and solve these types of decision models (an extensive survey of the literature in this field is contained in Van Delft and Nijkamp [1977] and Nijkamp [1977b]). A central role in multi-objective optimization theory is played by the concept of a Pareto solution (or non-inferior, efficient or non-dominated solution). A Pareto solution reflects the common feature of multi-objective optimization models that the value of the one objective function cannot be improved without affecting the values of the remaining objective functions. Such a solution shows the conflicting nature of these models: any feasible point that is not dominated by other points can be regarded as a Pareto solution.

In formal terms a Pareto solution can be defined as follows: a Pareto solution is a vector $\underline{x}^*$ for which no other feasible solution vector $\underline{x}$ does exist such that:

$$(2.2.) \quad \begin{cases} \underline{\omega}(\underline{x}) \geq \underline{\omega}(\underline{x}^*) \\ \text{and} \\ \underline{\omega}_j(\underline{x}) \neq \underline{\omega}_j(\underline{x}^*) \end{cases} \quad , \text{ for at least one } j$$

It has been proved among others by Geoffrion [1968] and Kuhn and Tucker [1968], that a feasible solution is a Pareto solution $\underline{x}^*$, if and only if a vector of weights $\underline{\lambda}$ does exist (with $\sum \lambda_i = 1$ and $\lambda_i \geq 0$), such that $\underline{x}^*$ is the optimal solution of the following uni-dimensional program:

$$(2.3.) \quad \begin{cases} \max \pi = \underline{\lambda}' \{ \underline{\omega}(\underline{x}) \} \\ \underline{x} \in K \end{cases}$$

By means of a parametrisation of $\underline{\lambda}$ the whole set of Pareto solutions can, in principle, be determined, although in practice the algorithms for determining this set appear to be rather time-consuming. Since the vector $\underline{\lambda}$ is a set of weights associated with each Pareto solution, it plays an important role in determining an ultimate equilibrium or compromise solution of a multi-objective model, particularly because any good solution of a multi-objective decision model should be a Pareto solution.

A figurative representation of the set of Pareto solutions (the efficiency frontier) is contained in Fig. 1, based on 2 objective functions.

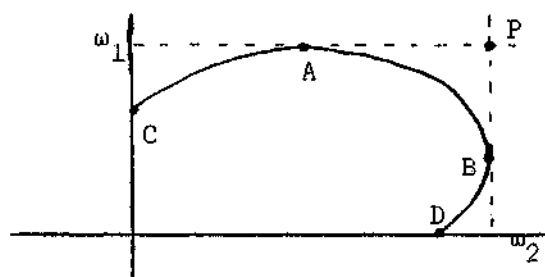


Fig. 1. A functional space with the efficiency frontier of 2 objectives.

A closer examination of Fig. 1 leads to the conclusion that only the points on the edge between A and B are relevant Pareto points, because (1) all interior points are dominated by the points on the edge, (2) all points on the edges CA and DB are dominated by point A and B, respectively, and (3) no point on the edge AB dominates any other point on this edge.

Point P of Fig. 1 can be regarded as the ideal point, which is used as a reference point for evaluating the points on the efficiency frontier. One may assume that the ultimate equilibrium (compromise) solution is that point which has a minimum discrepancy with respect to P. This minimum discrepancy can be measured by means of a Minkowski metric ψ . This gives rise to the following compromise model:

$$(2.4.) \quad \begin{cases} \min \psi = \left\{ \sum_{j=1}^J (1 - \hat{\omega}_j)^p \right\}^{1/p} \\ \hat{\omega}_j = \frac{\omega_j(\underline{x}) - \omega_j^{\min}}{\omega_j^{\max} - \omega_j^{\min}} \\ \underline{x} \in K \end{cases},$$

where $\hat{\omega}_j$ is the standardized value of objective function ω_j .

The solution of this compromise model can be calculated by applying non-linear programming techniques (see section 4).

In many decision procedures, however, this first compromise is not regarded as the final equilibrium solution, so that a certain interactive learning procedure has to be developed in order to reach in a series of steps such a final solution. Thus the provisional solution has to be presented to the decision-maker as a trial solution which has to be judged by him. The decision-maker has to indicate which objective functions are to be improved and which give already satisfactory results.

Let us denote now the set of objective functions which are to be increased in value by S, so that the decision-maker's preferences can be taken into account by specifying the following constraint:

$$(2.5.) \quad \omega_j(\underline{x}) \geq \bar{\omega}_j(\underline{x}) \quad \forall j \in S$$

In consequence, the following model has to be solved:

$$(2.6.) \quad \begin{cases} \max \omega(\underline{x}) \\ \underline{x} \in K \\ \omega_j(\underline{x}) \geq \bar{\omega}_j(\underline{x}) \quad \forall j \in S \end{cases}$$

Given this model, a new ideal point P_1 can be calculated. Clearly this displacement of the ideal point is due to condition (2.5.). After the calculation of the displaced ideal point, a new compromise solution can be determined by means of (2.4.) etc., until finally a satisficing compromise is attained.

This method of displaced ideals was originally developed by Zeleny [1976] and can be regarded as one of the most practible interactive multi-objective decision techniques. An empirical application of this technique to a regional industrialization problem can be found in Van Delft and Nijkamp [1977]. The latter method will also be employed in subsequent sections which focus on the use of non-linear multi-objective models in interdependent decision-making.

3. Generalized Non-linear Multi-objective Models.

Non-linear programming problems have been discussed extensively in mathematics and operations research (see for example Zangwill [1969]). The necessary and sufficient conditions for a global maximum of any differentiable objective function constrained by (in)equalities were derived by Kuhn and Tucker [1968] by means of Lagrangian theory. The general conclusion of Kuhn and Tucker is that a concavity of the objective function maximized within a set of convex constraints will guarantee a global maximum. In spite of the generality of their conclusion, the majority of programming models is still based on linear relationships. A specific type of non-linear programming models, viz. geometric programming, has received much attention during the last decade (see for an introduction Duffin et al. [1967] and Nijkamp [1972]). This method appears to fit quite well into the class of problems discussed in the previous section, in which objective functions (essentially, discrepancy measures) of a power-type were introduced. Therefore, a brief survey of geometric programming theory will first be given, based on a new presentation by means of a matrix formulation.

The primal formulation of a geometric programming model is based on a minimization of a posynomial (a sum of positive power functions) subject to a constraint set of posynomials. The general specification of such a model is:

$$(3.1.) \quad \begin{cases} \min \phi = c_0' f_0 \\ \text{subject to (s.t.)} \\ c_j' f_j \leq 1, & j = 1, \dots, J, \end{cases}$$

where:

$$(3.2.) \quad \underline{c}_j = (c_{j1}, \dots, c_{jI})' \geq 0, \quad j = 0, 1, \dots, J$$

and:

$$(3.3.) \quad \ln \underline{f}_j = A_j \ln \underline{x}, \quad j = 0, 1, \dots, J,$$

in which $\underline{f}_j$ is a vector of order $(I \times 1)$ with typical elements f_{ji} , A_j an $I \times K$ matrix with typical elements a_{ik}^j ($i=1, \dots, I$; $k=1, \dots, K$) and $\underline{x}$ a $(K \times 1)$ vector of decision variables (arguments). The coefficients a_{ik}^j may be positive or negative.

The dual specification of such a primal geometric model is:

$$(3.4.) \quad \begin{cases} \max \omega = \sum_{j=0}^J \underline{v}_j' (\ln \underline{c}_j - \ln \underline{v}_j) + \underline{u}' (\ln \underline{u}) \\ \text{subject to} \\ A' \underline{v} = \underline{e}_1 \\ \underline{v} \geq 0 \end{cases}$$

where $\underline{v}_j$ is a vector of dual variables v_i^j ($j=0, \dots, J$; $i=1, \dots, I$); each separate term of the posynomial program (3.1.) is related to its corresponding specific shadow variable. The remaining vectors are defined as:

$$(3.5.) \quad \begin{aligned} \underline{u} &= (u_1, \dots, u_J)' \\ &= (\underline{1}' \underline{v}_1, \underline{1}' \underline{v}_2, \dots, \underline{1}' \underline{v}_J)' \end{aligned}$$

with $\underline{1}$ being a vector of unit elements, and:

$$(3.6.) \quad \underline{e}_1 = (1, 0, \dots, 0)',$$

where $\underline{e}_1$ is of order $K+1$. Finally, matrix A' of order $(K+1) \times I(J+1)$ has the following structure:

$$(3.7.) \quad A' = \begin{bmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & & & \vdots \\ A'_0 & & A'_1 & \dots & A'_J \end{bmatrix}$$

The primal-dual relationships of the optimum solution of a geometric programming model are:

$$(3.8.) \quad v_i^0 = c_{oi} f_{oi} / \phi, \quad i=1, \dots, I$$

and

$$(3.9.) \quad v_i^j = c_{ji} f_{ji} u_j / \phi, \quad i=1, \dots, I; j=1, \dots, J$$

In this case the relationship between the primal and dual objective function is:

$$(3.10.) \quad e^{\omega} = \phi$$

Given the primal-dual relationships (3.8.) and (3.9.) and the duality condition (3.10.), there is a unique relationship between a primal and a dual geometric programming model. Proofs of the uniqueness of the solution can be found among others in Duffin et al. [1967], Lupták [1977], Nijkamp [1972] and Peterson [1976].

The foregoing geometric programming model was based on the existence of a single objective function. Now the question arises whether this assumption can be relaxed, so that multiple objective functions can also be taken into account.

A first method to deal with multi-objective geometric programming models is to assume a set of primal objective functions ϕ , i.e.,

$$(3.11.) \quad \underline{\phi} = (\phi_1, \dots, \phi_N)',$$

which are to be minimized within the constraint set already specified in (3.1.). It has often been shown (see among others Geoffrion [1967], Kuhn and Tucker [1968], and Vemuri [1974]), that such multi-objective programs can be numerically solved if a parametric programming algorithm is available for the following (constrained) linear convex combination of the N objective functions:

$$(3.12.) \quad \tilde{\phi} = \underline{\alpha}' \underline{\phi},$$

where $\underline{\alpha}$ is a weight vector satisfying the additivity condition:

$$(3.13.) \quad \underline{1}' \underline{\alpha} = 1$$

This approach is in fact a straightforward application of non-linear programming theory and gives rise to the well-known set of Pareto solutions discussed already in section 2.

An alternative and new approach to multi-objective geometric programs might be to define the parametric program not as a linear combination of the original N objective functions (an arithmetic mean), but as a non-linear combination via a power function (a geometric mean). Then the new parametric objective function reads as:

$$(3.14.) \quad \min \tilde{\phi} = \prod_{n=1}^N \phi_n^{\alpha_n}$$

or

$$(3.15.) \quad \min \ln \hat{\phi} = \sum \alpha_n' \ln \phi_n$$

It should be noted that $\hat{\phi}$ is not necessarily a posynomial expression, but it can easily be transformed into a posynomial expression by means of an auxiliary posynomial constraint for ϕ_n . Therefore, the generalized parametric representation of a vector-valued (multi-objective) geometric programming model is:

$$(3.16.) \quad \begin{cases} \min \hat{\phi} \\ \text{subject to} \\ c_j' f_j \leq 1 \\ \ln \hat{\phi} = \sum \alpha_n' \ln \phi_n \\ c_{on}' f_{on} \phi_n^{-1} = 1 \quad \forall n \end{cases}$$

The advantage of this specification is that the parameters α_n can be interpreted as elasticities of the individual objectives ϕ_n with respect to the 'master' objective function $\hat{\phi}$. Furthermore, the degrees of freedom in the dual program (i.e. the number of independent variables over which the dual program is to be maximized) of $\hat{\phi}$ is smaller than that of ϕ , so that the first one is easier to solve. It is clear that due to the arithmetic-geometric inequality conditions (see Hardy et al [1959]) the following inequality is valid:

$$(3.17) \quad \phi \geq \hat{\phi},$$

with an equality, for any value of α_n , if and only if $\phi_1 = \phi_2 = \dots = \phi_N$.

It has been shown by Pascual and Ben-Israel [1971] that the optimal solutions of ϕ and $\hat{\phi}$ are normally not the same, so that the optimal solutions of $\hat{\phi}$ need not be properly efficient solutions of the multi-objective geometric programming model. However, these solutions can be proved to be efficient. It is also obvious, that the usual features of geometric programming models hold also for (3.16.). Signomial problems (negative signs for certain terms) can also be incorporated in (3.16.), for example, in the case of minimizing and maximizing objective functions. In the latter case, the maximizing objective functions are provided with a negative exponent ($-\alpha_n$). The advantage of the power specification of (3.14.) is that in this case the global optimality of the equilibrium solution is still guaranteed, because a negative exponent does still give rise to a posynomial expression. This is also a considerable advantage compared to the linear parametric program (3.12.); in that case, a negative sign of α_n would lead to rather complicated signomial problems, for which a global optimality of a solution cannot be assumed.

Computational aspects of polynomial and signomial problems will not be discussed here, but are dealt with among others in Dinkel et al. [1974], Duffin and Peterson [1973], Nijkamp [1972] and Rijckaert [1977].

4. A Simple Conflict Model for Growth and Environmental Quality.

Assume a city or region with 2 interest groupings with conflicting objectives, viz. supporters of economic growth and environmentalists. This situation will now be described by means of a simple model.

The supporters of economic growth want to maximize production q , i.e.

$$(4.1.) \quad \max \phi_1 = q$$

The production is assumed to be related to productive investments i by means of a non-linear technological function incorporating scale advantages, i.e.,

$$(4.2.) \quad q = \alpha i^\beta, \quad \beta > 1$$

The environmentalists aim at maximizing environmental quality e , i.e.,

$$(4.3.) \quad \max \phi_2 = e$$

This objective function can be operationalized by assuming that environmental quality can be improved by spending a large part of available resources to the preservation of environmental commodities (such as abatement investments) and to the creation of natural areas. By denoting these environmental investments by z , the second objective function may now be written as:

$$(4.4.) \quad \max \phi_2 = \theta z^\mu,$$

where the assumption is made that environmental quality is related in the following way to environmental investments:

$$(4.5.) \quad e = \theta z^\mu, \quad \mu \leq 1$$

The latter relationships indicates that every decrease in environmental quality (caused by an increase in production or consumption) may be compensated by environmental investments. The meaning of this assumption can be illustrated by assuming the existence of a certain pollution emission relationship (see Nijkamp [1977a]). The pollution is assumed to be related to the production by means of a non-linear emission function:

$$(4.6.) \quad p = \gamma q^\delta, \quad \delta \leq 1$$

Next, the assumption is made that the emission coefficient γ can be influenced by implementing abatement investments z , i.e.,

$$(4.7.) \quad \gamma = \eta z^{-\epsilon}$$

Substitution of (4.6.) into (4.5.) yields the result:

$$(4.8.) \quad p = \eta z^{-\epsilon} q^{\delta},$$

so that any increase in the emission of pollution can be reduced by implementing more abatement investments.

Clearly, the maximum of (4.1.) would be infinite, if there would be no constraints on the investments. Similarly, the maximum of ϕ_2 would be infinite, if there would be no constraints on the available resources. Therefore, it is plausible to assume an upper limit t for the total investment budget which may be allocated between productive investments and environmental investments, i.e.,

$$(4.9.) \quad i + z \leq t$$

The variables i and z may be regarded as the decision variables which determine the value of the arguments of the urban (or regional) welfare profile, viz. q and e . The decision space is represented by Fig. 2.

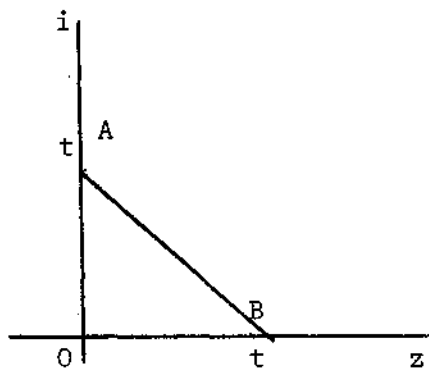


Fig. 2. Decision space of productive and abatement investments.

It is clear that a maximization of (4.1.) subject to (4.9.) will give A as the optimal solution. The maximization of (4.3.) subject to (4.9.) obviously leads to B as the optimal solution; the compromise may be located somewhere on AB.

A compromise solution between maximum growth and maximum environmental quality can be formally found by using the idea of a multi-objective programming analysis set out in section 2 and 3.

By applying the idea of a geometric parametrisation of the objective functions (see (3.14.)), the multi-objective geometric programming model

associated with the abovementioned model can be specified as:

$$(4.10.) \quad \begin{cases} \max \phi = \phi_1^{\lambda_1} \phi_2^{\lambda_2} \\ \text{s.t. (4.9.)} \end{cases},$$

where λ_1 and λ_2 are the parametric weights (elasticities). The latter objective model can be re-written in a standard geometric programming format as:

$$(4.11.) \quad \begin{aligned} \min \dot{\phi} &= \phi^{-1} = \phi_1^{-\lambda_1} \phi_2^{-\lambda_2} \\ &= \alpha^{-\lambda_1} i^{-\beta\lambda_1} \theta^{-\lambda_2} z^{-\mu\lambda_2} \\ &= \alpha^* i^{-\beta\lambda_1} z^{-\kappa\lambda_2}, \end{aligned}$$

where:

$$(4.12.) \quad \alpha^* = \alpha^{-\lambda_1} \theta^{-\lambda_2}$$

Obviously, a high value of i and z will lead to a low value of $\dot{\phi}$. Finally, the following model can be obtained:

$$(4.13.) \quad \begin{cases} \min \dot{\phi} = \alpha^* i^{-\beta\lambda_1} z^{-\kappa\lambda_2} \\ \text{s.t.} \\ t^{-1} i + t^{-1} z \leq 1 \end{cases},$$

which corresponds to the general geometric programming model (3.1.).

According to (3.7.) the dual constraints of this model can be written as:

$$(4.14.) \quad \begin{bmatrix} 1 & 0 & 0 \\ -\beta\lambda_1 & 1 & 0 \\ -\kappa\lambda_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^0 \\ v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The number of degrees of freedom of these dual constraints appears to be equal to 0, so that the dual variables can be directly solved from (4.14.) by means of a simple matrix inversion. This leads to the following result:

$$(4.15.) \quad \begin{bmatrix} v_1^0 \\ v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\beta\lambda_1 & 1 & 0 \\ -\kappa\lambda_2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \beta\lambda_1 \\ \kappa\lambda_2 \end{bmatrix}$$

According to the primal-dual relationships (see section 3) the following optimal solutions of the variables i and z can be derived:

$$(4.16.) \quad i^0 = \frac{\beta \lambda_1}{\beta \lambda_1 + \kappa \lambda_2} t$$

and

$$(4.17.) \quad z^0 = \frac{\kappa \lambda_2}{\beta \lambda_1 + \kappa \lambda_2} t$$

By means of a parametrisation of (4.10.), the efficiency frontier of the objective functions ϕ_1 and ϕ_2 can, in principle, be determined (see Fig. 3; see also section 2).

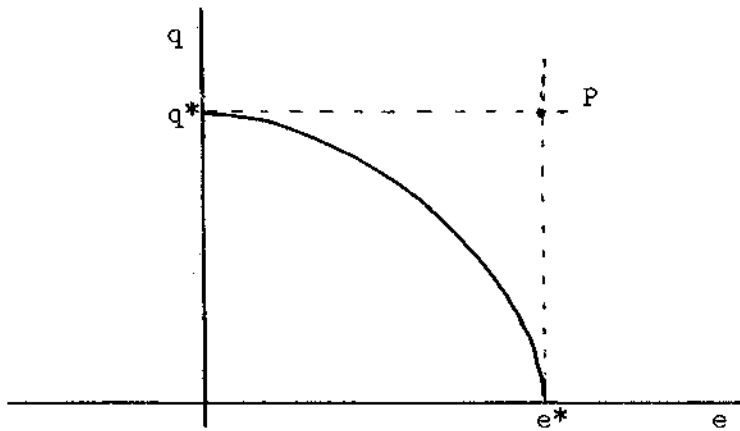


Fig. 3. The efficiency frontier of q and e .

This figure demonstrates that by means of new multi-objective programming techniques, in principle, an ultimate compromise solution can be identified.

The first conclusion is thus that non-linear optimization models describing conflicts between diverging objectives can be attacked by means of generalized multi-objective posynomial models.

The ideal point P of Fig. 3 (with co-ordinates q^* and e^*) can now be used as a frame of reference for an interactive approach based on a successive series of 'displaced ideals'. This implies that a trial solution has to be identified which is calculated by means of a Minkowski metric for a minimum discrepancy between the ideal point and the efficiency frontier. Therefore, the following program has to be solved (see also (2.4.)):

$$(4.18.) \left\{ \begin{array}{l} \min \psi = \{ (1 - \pi_q)^v + (1 - \pi_e)^v \}^{1/v} \\ \text{s.t.} \\ \pi_q = \frac{\phi_1 - \phi_1^{\min}}{\phi_1^{\max} - \phi_1^{\min}} \\ \pi_e = \frac{\phi_2 - \phi_2^{\min}}{\phi_2^{\max} - \phi_2^{\min}} \\ \phi_1 = \alpha i^\beta \\ \phi_2 = \theta z^\kappa \\ i + z \leq t \end{array} \right.,$$

where $\phi_1^{\min}$ and $\phi_2^{\min}$ are the minimum feasible values of ϕ_1 and ϕ_2 , respectively, and $\phi_1^{\max}$ ($=q^*$) and $\phi_2^{\max}$ ($=e^*$) the maximum feasible values of ϕ_1 and ϕ_2 , respectively.

The latter program is again a geometric program, as can easily be seen by rewriting (4.18.) as:

$$(4.19.) \left\{ \begin{array}{l} \min \psi = r^{1/v} \\ r^{-1} r_q^v + r^{-1} r_e^v = 1 \\ r_q + \pi_q = 1 \\ r_e + \pi_e = 1 \\ (\phi_1^{\max} - \phi_1^{\min}) \pi_q \phi_1^{-1} + \phi_1^{\min} \phi_1^{-1} = 1 \\ (\phi_2^{\max} - \phi_2^{\min}) \pi_e \phi_2^{-1} + \phi_2^{\min} \phi_2^{-1} = 1 \\ t^{-1} i + t^{-1} z \leq 1 \\ \phi_1 = \alpha i^\beta \\ \phi_2 = \theta z^\mu \end{array} \right.$$

The latter program can be solved by means of standard geometric (posynomial) programming techniques. The solution of this program is a point somewhere on the efficiency frontier of Fig. 3, and will be denoted by $\hat{q}_1$ and $\hat{e}_1$. This trial solution may be used as a tool in an interactive urban or regional decision-making process: the only information needed concerns the question which value of the trial solutions is not satisfactory. This gives rise to a new constraint which may be added to (4.13.) (see also section 2). Consequently, a (horizontal or vertical) displacement of the ideal solution P toward the axis of the satisfactory solution takes place. Then the procedure may be repeated again and

again, until finally a converging compromise solution is attained.

The second conclusion is that geometric programming models may be useful tools for interactive decision models based on the method of displaced ideals via a Minkowski metric.

5. A Spatial Externalities Model for Displaced Ideals.

Finally, interdependences in a spatial system can also be studied in a more appropriate manner by means of the foregoing multi-objective posynomial approach. A first way of introducing spatial interdependences is to assume a set of cities or regions with negative mutual spill-over effects. An example may be the existence of environmental externalities, so that the environmental quality of region 1 is affected by the pollution resulting from the production of region 2. This problem can be regarded as a straightforward generalization of the foregoing conflict model and can in principle be solved by means of the same approach via displaced ideals (see also Hafkamp and Nijkamp [1978]). This extension will not be dealt with here any further.

An alternative assumption may be a 'keeping up with the Joneses' effect. This implies that each city or region within the spatial system at hand evaluates its welfare on the basis of a reference profile which is co-determined by the welfare levels of other regions (see also Klaassen [1978]).

The basic idea is here that the welfare of each city or region r ($r = 1, \dots, R$) can be represented by means of a welfare profile $\underline{x}_r$ (see section 2) with arguments x_{ir} ($i=1, \dots, I; r=1, \dots, R$). Without loss of generality the assumption is made that for each element of the welfare profile the condition holds: 'the higher, the better', so that the multi-dimensional objective function of each city or region is:

$$(5.1.) \quad \max \phi_r = \underline{x}_r$$

This multi-objective model can again be solved by means of the approach set out above.

In the case of spatially interdependent behaviour one may assume that each region or city evaluates its welfare profile against the background of the maximum attainable welfare profile in the whole spatial system concerned. In other words, the reference profile $\underline{x}^{\max}$ has the following elements $x_i^{\max}$:

$$(5.2.) \quad x_i^{\max} = \max_r x_{ir}, \quad \forall i$$

Therefore, the i^{th} objective function of each spatial unit or region may be assumed to be:

$$(5.3.) \quad \min \psi_{ir} = x_i^{\max} - x_{ir} \quad 1)$$

The latter multi-objective optimization problem can also be attacked by means of the method described in section 4, as will be illustrated now.

Let us first assume a multi-objective multi-regional, but spatially independent optimization problem (see (5.1.)). In case of two regions 1 and 2 the model from section 4 can be written as:

$$(5.4.) \quad \begin{cases} \max \phi_{11} = q_1 \\ \max \phi_{12} = e_1 \\ \max \phi_{21} = q_2 \\ \max \phi_{22} = e_2 \\ \text{s.t. (4.2.), (4.4.) and (4.9.)} \end{cases}$$

By applying a geometric parametrization of this multi-objective model the following optimization problem can be specified (see (4.13.)):

$$(5.5.) \quad \begin{cases} \min \phi = \alpha^{**} i_1^{-\beta_1 \lambda_{11}} z_1^{-\kappa_1 \lambda_{12}} i_2^{-\beta_2 \lambda_{21}} z_2^{-\kappa_2 \lambda_{22}} \\ \text{s.t.} \\ t_1^{-1} i_1 + t_1^{-1} z_1 \leq 1 \\ t_2^{-1} i_2 + t_2^{-1} z_2 \leq 1 \end{cases}$$

The dual constraints of this geometric programming model are:

$$(5.5.) \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\beta_1 \lambda_{11} & 1 & 0 & 0 & 0 \\ -\kappa_1 \lambda_{12} & 0 & 1 & 0 & 0 \\ -\beta_2 \lambda_{21} & 0 & 0 & 1 & 0 \\ -\kappa_2 \lambda_{22} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^0 \\ v_1^1 \\ v_1^1 \\ v_2^1 \\ v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so that the dual variables are equal to:

- 1) Clearly, if $x_{ir} = x_i^{\max}$, the objective function should be $\max x_{ir}$.

$$(5.6.) \quad \begin{bmatrix} v_1^0 \\ v_1^1 \\ v_2^1 \\ v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \beta_1 \lambda_{11} \\ \kappa_1 \lambda_{12} \\ \beta_2 \lambda_{21} \\ \kappa_2 \lambda_{22} \end{bmatrix}$$

Consequently, the optimal values of the decision variables are:

$$(5.7.) \quad \begin{aligned} i_1^0 &= \frac{\beta_1 \lambda_{11}}{\beta_1 \lambda_{11} + \kappa_1 \lambda_{12}} t_1 & z_1^0 &= \frac{\kappa_1 \lambda_{12}}{\beta_1 \lambda_{11} + \kappa_1 \lambda_{12}} t_1 \\ i_2^0 &= \frac{\beta_2 \lambda_{21}}{\beta_2 \lambda_{21} + \kappa_2 \lambda_{22}} t_2 & z_2^0 &= \frac{\kappa_2 \lambda_{22}}{\beta_2 \lambda_{21} + \kappa_2 \lambda_{22}} t_2 \end{aligned}$$

The procedure of displaced ideals via a Minkowski metric discussed in section 4 can be applied here in an analogous manner.

Now the assumption of a multi-objective multi-regional spatially inter-dependent decision system may be made, so that an objective function of type (5.2.) has to be used. Then the following multi-objective function may be specified:

$$(5.8.) \quad \min \psi = \prod_{i,r} (x_i^{\max} - x_{ir})^{\lambda_{ir}}$$

or in terms of the foregoing model:

$$(5.9.) \quad \begin{cases} \min \psi = (q^{\max} - q_1)^{\lambda_{11}} (e^{\max} - e_1)^{\lambda_{12}} (q^{\max} - q_2)^{\lambda_{21}} (e^{\max} - e_2)^{\lambda_{22}} \\ \text{s.t. (4.2.), (4.4.) and (4.9.)} \end{cases}$$

The latter model can be solved by means of standard geometric programming techniques via the abovementioned procedure of displaced ideals. The outcome is in fact the result of a game strategy with multiple participants.

The latter model provides a static picture. Given the reference pattern of a maximum welfare profile, the compromise solution for each city or region can be identified via a parametric multi-objective procedure. It should be noted, however, that after a certain period the reference pattern undergoes a change due to the decisions in each city or region. This implies that a realization of the optimal values of the decision variables exerts an influence upon the reference pattern for the decision in a next period. This shift in $x^{\max}$ can also be regarded as a displacement of ideals in a dynamic setting. A further analysis of the equilibrium conditions of such a recursive procedure of displaced ideals would require the use of a complete dynamic model which might, for example, be solved by means

of Bellman's optimality principle. The essential ideas of displaced ideals in a multi-regional welfare setting, however, will remain the same in such a multi-objective programming model. The main difference concerns the shifts in urban (regional) resources due to the successive production and investment decisions in previous periods and the resulting shifts in ideal points.

The conclusion of this section is that non-linear multi-objective decision models are a useful tool for adaptive choice behaviour leading to interdependent decisions between actors.

6. Conclusion.

Modern optimization theory is, in many respects, confronted with conflicting objectives, among which a compromise has to be found. Multi-objective decision theory can be regarded as an appropriate tool to treat these problems. Especially the method of displaced ideals appears to be meaningful to find a compromise solution in an interactive manner. The use of geometric programming techniques is useful to deal with non-linearities in these types of models. Finally, interdependent decision-making via a recursive shift of ideal reference patterns can also be dealt with in an analogous manner; this gives rise to a generalization of the ideas of displaced ideals over a series of decision periods. Consequently, the notion of a displaced ideal is an extremely important concept for a more advanced optimization methodology for multi-regional planning and decision-making.

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