

GRAVITY AND ENTROPY MODELS:

THE STATE OF THE ART.

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"Man tends, of necessity, to gravitate towards his fellow man"

(H.C. Carey, Principles of Social Science, Lippincott, Philadelphia, 1858, p. 42).

1. Introduction

The explanation of a spatial pattern of activities has been a major focus of location theory for a long time. During the last decade, however, the attention for a spatial structure per se has been broadened by taking into account the configuration of spatial processes as well. The popularity of spatial interaction models reflects the importance attached to spatial process models in spatial system design (see for example Masser and Brown [1977]). The "geography of movement" appears to open a rich field of new scientific research (cf. Lowe and Moryades [1975]). Spatial interaction theory has become an important descriptive device for the dispersion and coherence of activities in a spatial system. The major part of spatial interaction models derives its theoretical perspective from gravity or entropy theory.

Therefore, in this paper attention will be paid to the foundations of gravity and entropy theory. A brief survey of both gravity theory and entropy theory will be given, followed by an evaluation and critical review of these theories. Particular attention will be devoted to the methodological problem of interpreting these theories as behavioural devices in a social science. An attempt will be made to derive a new interpretation of entropy theory via the use of multi-objective optimization models. In addition, a method will be proposed to include prior knowledge on spatial processes in a behavioural framework of a generalized entropy model via a Bayesian type approach. Finally, the use of entropy models in an empirical context, especially in the field of spatial interaction and disaggregation, will be discussed.

2. Gravity Models

Gravity theory describes the degree of spatial interaction between two or more points in a spatial system in a manner analogous to point-to-point movements between elements in a physical system. The crucial element in traditional gravity theory is formed by Newton's law from physics, which states that the attraction force a_{ij} between 2 entities i and j is proportional to their respective masses m_i and m_j and inversely proportional to the squared distance s_{ij}^2 between these entities. Consequently, this law of gravitational attraction states:

$$(2.1.) \quad a_{ij} = \gamma m_i m_j s_{ij}^{-2},$$

where γ is a constant proportionality factor.

The traditional gravity model in spatial interaction theory is essentially based on a spatial analogy with regard to Newtonian physics. In many spatial flow models it has been found that distance between two points in space and the size of their populations (mass) may provide a reasonably good statistical explanation for the volumes of these flows.

The first attempt to apply some concepts from Newtonian physics to spatial interaction flows was made by Ravenstein [1885] in a study of migration flows between English cities. Ravenstein's "law on migration" was in fact based on a simplified law of Newton, because he explained migration flows m_{ij} from i to j via the population size p_j ("potential") in the centre of destination and the distance s_{ij} between i and j . Other earlier contributions to simplified gravity models were presented by Reilly [1931] and Young [1924] in the field of trade flows and farm population migrations, respectively (see for a survey also Carrothers [1954]).

First complete specifications of spatial analogies of Newton's law were presented by Stewart [1941] and Zipf [1949]. Their models had the following form:

$$(2.2.) \quad t_{ij} = \gamma p_i p_j s_{ij}^{-2},$$

where t_{ij} is the degree of spatial interaction from i to j (for example, migration flows, telephone calls, trade flows, passenger traffic etc.) The proportionality constant γ may be regarded as a scalar which adjusts the model to the units of measurement of the variables. Apart from dimensional reasons, this constant was interpreted by Stewart as the total energy of interaction of i with respect to all other points j . This problem will be discussed later.

It is clear that there is no theoretical need to include the exponent 2 as a spatial discount factor in the distance friction function, so that in later specifications of gravity models frequently the following function has been used:

$$(2.3.) \quad t_{ij} = \gamma p_i p_j s_{ij}^{-\beta}, \quad \beta > 0$$

A more general specification may even be a Cobb-Douglas type of gravity model:

$$(2.4.) \quad t_{ij} = \gamma p_i^\alpha p_j^\epsilon s_{ij}^{-\beta}$$

The difference between (2.3.) and (2.4.) is that in the former relationship t_{ij} is calculated as an unweighted geometric average of p_i and p_j , whereas

in (2.4.) a weighted geometric average is used (see also Hardy et al. [1967]). It should be noted that the inequality relationship between the geometric average and the normal arithmetic average is one of the points of departure for the theory of geometric programming (cf. Duffin et al. [1967] and Nijkamp [1972]). This theory will be discussed later, because it plays a crucial role in a behavioural interpretation of entropy models.

First, however, the methodological value of the use and application of "social physics" will be examined more closely. During the last decades the use and application of "social physics" has been severely criticized for several reasons such as the lack of theoretical justification, the mechanistic basis of gravity theory, and the simplification of social reality by means of physical gravity models.

In our opinion, several methodological requirements may be specified which are to be fulfilled in order to justify the use of methods and theories from physics in social sciences (cf. also Crilly [1974] and Heggie [1969]).

- (1) The method should satisfy normal logical conditions for the social system at hand (for example, spatial additivity conditions for the various flows).
- (2) There should be a high degree of direct correspondence between the phenomenon described in the physical world and that from a spatial system.
- (3) The physical basis of the approach should be interpreted in terms of social aspects of the system at hand (for example, in terms of social behavioural hypotheses or at least in terms of statistical regularities of aggregate spatial choices of the social system at hand).
- (4) The mathematical specification of the various relationships should be in correspondence with reasonable hypotheses about the behaviour of spatial interactors.
- (5) The empirical results of a physical approach should not contradict reality (or information about reality).

Condition (1) is not a priori guaranteed by a traditional gravity model, because the additivity condition (the sum of flows should add up to the total volume of the flows in the system) is not always satisfied. This condition implies:

$$(2.5.) \quad \sum_{j=1}^J t_{ij} = o_i \quad \forall i$$

and:

$$(2.6.) \quad \sum_{i=1}^I t_{ij} = d_j \quad \forall j$$

where o_i and d_j are the (known) volume of flows at the points of origin and of destination, respectively. When o_i and d_j are regarded as the respective masses at the origin and destination, then p_i and p_j from (2.3.) may be replaced by o_i and d_j . Now it is easily seen that the additivity conditions are only satisfied, if the gravity model is specified as:

$$(2.7.) \quad t_{ij} = a_i b_j o_i d_j s_{ij}^{-\beta},$$

where the balancing factors a_i and b_j are equal to: ¹⁾

$$(2.8.) \quad a_i = \left(\sum_{j=1}^J b_j d_j s_{ij}^{-\beta} \right)^{-1}$$

and:

$$(2.9.) \quad b_j = \left(\sum_{i=1}^I a_i o_i s_{ij}^{-\beta} \right)^{-1}$$

On the basis of information on t_{ij} , s_{ij} , o_i and d_j the parameters a_i , b_j and β can be assessed by means of recursive solution techniques (see for example, Batty [1970], Batty and Mackie [1973], Bouchard and Pyers [1964], Chisholm and O'Sullivan [1973], Hyman [1969] and Wilson et al. [1969]).²⁾ The balancing factors of such a doubly constrained spatial interaction model can be interpreted as a measure of the accessibility (in terms of attractiveness and repulsion) of one zone with respect to other zones (see Kirby [1970]).

Condition (2) is normally fulfilled because there is a strong formal analogy, if not a direct correspondence, between the movements of gas molecules in a physical system and the point-to-point movements in a spatial interaction system: in both systems the behaviour of elements is determined by push and pull factors in addition to distance frictions.

Condition (3) is hardly fulfilled, because there is a priori no logical

- 1) Clearly, without the additivity conditions a doubling of both all o_i 's and d_j 's would quadruple the volume of trips between all points i and j .
- 2) It is easily seen that in the case of model (2.4.) the balancing factors are:

$$a_i = o_j^{1-\alpha} \left(\sum_{j=1}^J b_j d_j s_{ij}^{-\beta} \right)^{-1}$$

and:

$$b_j = d_j^{1-\alpha} \left(\sum_{i=1}^I a_i o_i s_{ij}^{-\beta} \right)^{-1}$$

reason why people should behave like gas molecules: choice behaviour is an essential ingredient of social sciences and this can hardly be represented by means of a physical theory. Nevertheless, it may be possible to find a formal behavioural rationale for the use and specification of a gravity model (see section 3).

The fourth condition is closely linked to the third condition and will be re-examined in the next section.

Condition (5) appears to be fulfilled in many empirical applications. The gravity model appears to be a rather robust mirror of observed spatial configurations. It is even not far from reality to state that the popularity of gravity models is mainly due to its good empirical results (apart from its simplicity) and that the search for a theoretical foundation and justification has started only after numerous reasonable outcomes in practice.

In conclusion, gravity theory appears to fulfil conditions (1),(2) and (5). The fulfilment of conditions (3) and (4) is somewhat more doubtful. Therefore, in the next section an attempt will be made to examine whether it is possible to derive a gravity theory in which conditions (3) and (4) are also satisfied.

3. A Utility Foundation and Specification of Gravity Theory

It is clear that any utility foundation of gravity theory should be based on push and pull effects as well as on distance frictions between elements in space.

A first way to derive the gravity model is to abandon any physical analogy or physical basis and to derive a gravity specification on the basis of reasonable hypotheses about spatial interactions (see for example Isard [1960]). Assume a country with I cities i ($i=1, \dots, I$) and with a homogeneous population. The total size of population is p , while the population size of city i is denoted by p_i . The total number of interactions (trips, e.g.) between all cities i and j is assumed to be equal to t . Furthermore, all movements are assumed to be costless for the moment.

Given these assumptions, the trips made by an individual from city i to j depend on the size of i and j . It is clear that the expected percentage of trips of this individual to city j is equal to p_j/p . Furthermore, it is clear that the expected (average) number of trips made by any arbitrary individual is equal to t/p (due to the homogeneity assumption). This number can be set equal to a constant k . Therefore, the expected number of trips per individual in city i to j is equal to $k p_i p_j / p$. Now it is clear that the total number of expected trips from city i to j is:

$$(3.1.) \quad t_{ij}^* = k p_i p_j / p$$

Next, the assumption of a costless movement, and hence of a homogeneous population, may be relaxed, viz. by assuming that the ratio of actual to be expected trips (t_{ij}/t_{ij}^*) is an inverse function of distance:

$$(3.2.) \quad t_{ij}/t_{ij}^* = c s_{ij}^{-\beta}$$

Substitution of (3.1.) into (3.2.) gives:

$$(3.3.) \quad t_{ij} = \gamma p_i p_j s_{ij}^{-\beta},$$

where γ is equal to:

$$(3.4.) \quad \gamma = c t/p^2$$

Therefore, the gravity model may also be derived on the basis of rather simple principles for spatial interaction. It should be noted, however, that the underlying assumptions are rather rigid and contain some implicit behavioural hypotheses about spatial behaviour which should be made more explicit (see later).

An alternative foundation for a theory of spatial interaction behaviour based on a gravity-type approach was provided by Smith [1977]. By means of advanced topological notions he proposes a macro-statistical principle of spatial interaction, the so-called cost-efficiency principle. This principle asserts in essence that, other things being equal, spatial flow patterns involving higher total costs are less likely to be observed than those involving lower costs. This principle appears to characterize completely the class of gravity models.

Given the cost-efficiency principle as a macro-device for a spatial system, Smith shows even that from among all possible probabilistic models of trip behaviour which vary continuously with the underlying structure of travel costs, the only models which are consistent with the cost-efficiency principle are the exponential gravity models.

The proof of this statement runs via several steps, reflected by assumptions of a continuous probability distribution over trips as a function of travel costs, of trip independencies (i.e. absence of multi-purpose trips), of constant average travel costs, and of an inverse relationship between trip pattern probabilities and total travel costs of each trip pattern. Via a whole set of successive lemma's Smith shows ultimately that a trip distribution is cost-efficient if and only if there exists an exponential gravity representation for this trip pattern.

The relevance of this approach is that exponential gravity models are derived from macro-statistical principles which are more directly interpretable behaviourally. Smith's approach appears to satisfy the methodological conditions (1) (logic and consistency), (2) (high degree of formal analogy), (3) (behavioural interpretation), (4) (reasonable mathematical specification) and (5) (link to reality). It should be noted, however, that the cost efficiency principle is only a macro-statistical principle. This principle does not state that individuals tend to make less costly trips rather than more costly trips. In other words, micro utility-maximizing behaviour may be a foundation of the cost efficiency principle, but it has not been proved that micro-behaving units tend on the whole to minimize travel expenditures at each level of trip activity.

A second problem arises from the generality of the cost-efficiency principle. While it states that less costly trip configurations are more probable than more costly trip configurations at a given level of trip activity, it does not specify in which way at a micro or macro level different distance frictions, and push and pull effects are evaluated by the spatial interactors. In other words, the mathematical specification of a utility model for evaluating alternative trip costs is missing. The specification of such a utility model will now be discussed.

Thus the question arises whether a behavioural criterion can be specified which can be interpreted as a utility function associated with a gravity model. Earlier contributions to an analysis of relationships between utility theory and gravity theory are contained among others in Golob and Beckmann [1971], Golob et al. [1973], Niedercorn and Bechdolt [1969] and Nijkamp [1975].

Suppose that each point of origin i can be regarded as an aggregate decision unit for trip decisions in a spatial interaction framework. Consequently, the hypothesis may be made that this decision unit has an aggregate welfare function (utility function or cost function), which may be regarded as a representative average welfare function for trip behaviour of the individuals living in this point of origin. The spatial decision unit is next assumed to allocate a certain aggregate "travel budget" among a series of trips for alternative destinations. This travel budget is assumed to be a function of the number of individuals living at point i . Thus an optimal trip allocation is achieved by maximizing the above mentioned welfare function subject to the conditions arising from the travel budget.

Consequently, the utility-maximizing problem of point of origin i is:

$$(3.4.) \left\{ \begin{array}{l} \max w_i = f_i(t_{i1}, \dots, t_{iJ}) \\ \text{subject to} \\ \sum_{j=1}^J c_{ij} t_{ij} = c_i \end{array} \right. ,$$

where c_{ij} represents the unit transportation cost between i and j and c_i the total travel budget of all individuals at point i . Next, c_i is assumed to be related to the total number of trip-makers at point i by means of a power function:

$$(3.5.) \quad c_i = \eta o_i^\alpha,$$

which implies that the travel budget is assumed to be proportional (via the parameter η) to the total push effect (o_i^α) of place i . Substitution of (3.5.) into (3.4.) and application of the first-order maximizing procedure gives:

$$(3.6.) \quad \begin{aligned} w'_{ij} &= \partial w_i / \partial t_{ij} & \forall i, j \\ &= \lambda c_{ij}, \end{aligned}$$

where λ is the Lagrange multiplier related to the budget condition. For all unknown variables from point i the following system of equations can be constructed:

$$(3.7.) \quad \begin{cases} w'_{i1} / c_{i1} = w'_{iJ} / c_{iJ} \\ \vdots \\ w'_{i,J-1} / c_{i,J-1} = w'_{iJ} / c_{iJ} \\ \sum_{j=1}^J c_{ij} t_{ij} = \eta o_i^\alpha \end{cases}$$

Should it now be possible to cast the gravity model (2.4.) in such a form that it satisfies (3.7.), then it would be possible to find a behavioural interpretation of the gravity model.¹⁾ It is easily seen that (2.4.) can be rewritten as follows for point i :

$$(3.8.) \quad \begin{aligned} \eta o_i^\alpha &= \eta \gamma^{-1} t_{ij} d_j^{-\epsilon} s_{ij}^\beta, & \forall j \\ &= \sum_{j=1}^I c_{ij} t_{ij} \end{aligned}$$

Next, it can be directly checked that a sufficient condition for the fulfilment of condition (3.8.) is:

1) It should be noted that any monotonically increasing transformation of w_i would lead to the same first-order conditions as in (3.7.).

$$(3.9.) \quad \begin{cases} n \gamma^{-1} t_{ij} d_j^{-\epsilon} s_{ij}^{\beta} \left\{ d_1^{\epsilon} s_{i1}^{-\beta} \left(\sum_{j=1}^J d_j^{\epsilon} s_{ij}^{-\beta} \right)^{-1} \right\} = c_{i1} t_{i1} \\ \vdots \\ n \gamma^{-1} t_{ij} d_j^{-\epsilon} s_{ij}^{\beta} \left\{ d_j^{\epsilon} s_{ij}^{-\beta} \left(\sum_{j=1}^J d_j^{\epsilon} s_{ij}^{-\beta} \right)^{-1} \right\} = c_{ij} t_{ij} \\ \vdots \\ n \gamma^{-1} t_{ij} d_j^{-\epsilon} s_{ij}^{\beta} \left\{ d_J^{\epsilon} s_{iJ}^{-\beta} \left(\sum_{j=1}^J d_j^{\epsilon} s_{ij}^{-\beta} \right)^{-1} \right\} = c_{iJ} t_{iJ} \end{cases}$$

By dividing the first $J-1$ equations by the last equation one obtains:

$$(3.10) \quad \begin{cases} d_1^{\epsilon} s_{i1}^{-\beta} / c_{i1} t_{i1} = d_J^{\epsilon} s_{iJ}^{-\beta} / c_{iJ} t_{iJ} \\ \vdots \\ d_{J-1}^{\epsilon} s_{i,J-1}^{-\beta} / c_{i,J-1} t_{i,J-1} = d_J^{\epsilon} s_{iJ}^{-\beta} / c_{iJ} t_{iJ} \end{cases}$$

Now it is evident that (3.10) can be regarded as a specification of the first-order maximizing conditions (3.7.), when the welfare function w_i of point i has the following shape:

$$(3.11.) \quad w_i = d_1^{\epsilon} s_{i1}^{-\beta} \ln t_{i1} + \dots + d_J^{\epsilon} s_{iJ}^{-\beta} \ln t_{iJ}$$

This log-linear welfare function of the trip decisions satisfies also the second-order maximizing conditions. This welfare function includes the elements $d_j^{\epsilon} s_{ij}^{-\beta}$ (V_j) as preference elasticities with respect to trip decisions. In other words, the pull effect of a point of destination j (i.e., d_j^{ϵ}) times the distance friction effect between i and j (i.e., $s_{ij}^{-\beta}$) constitute the preference weights of the aggregate trip utility function of point i . This result can be regarded as a reasonable and plausible utility interpretation of the gravity model.

Consequently, one may conclude that the use of a gravity model can be justified on the basis of an appropriate utility theory for aggregate trip behaviour. The implicit utility function appears to be a log-linear utility function with pull effects and distance friction effects as preference elasticities (although it should be noted that this function is not necessarily the only one which satisfies the first-order maximizing conditions for utility maximization due to the fact that only sufficient conditions were specified). Further discussions on the derivation of gravity-type trip distribution models can be found in Choukroun [1975].

4. Entropy Theory

From 1970 onward a new stimulus to the application of gravity-type models in a spatial interaction framework was given thanks to the rising popularity of entropy-maximizing models (see Wilson [1970]). The innovative significance of

entropy theory is that it places the traditional gravity-type interaction model in a probabilistic framework of a formal systems theory¹⁾.

The entropy concept stems originally from thermodynamics (see among others Fast [1970] and Mogridge [1972]). Closed physical systems (like a quantity of gas molecules) can in general adopt numerous states (i.e., alternative configuration of the elements of the system). It appears, however, that normally the elements of a closed physical system tend toward an arrangement which can be organized in as many ways as possible ('maximum disorder'). Such a tendency implies that a disordered configuration is much more probable than an ordered situation and is called a maximization of the entropy of the system.

The most likely state of a system is that particular state for which the number of combinatorial possibilities is the greatest. In other words, a closed physical system tends to adopt an equilibrium state defined as the most likely macro-state (or mega-state) tied up with such an arrangement of individual elements that it gives rise to a maximum number of feasible micro-states.

The entropy measure can be used to characterize both living and non-living systems. Especially in ecological systems the concept of entropy is gaining more importance, because it can be used as a device to analyse (dis)equilibria in ecosystems. In this respect, entropy is defined as the extent of unavailable energy during a series of normally irreversible processes in an ecosystem (see for a survey Nijkamp [1977] and Odum [1971]). In general, living systems tend to be open, so that living systems are capable to resist the trend toward disorder by capturing additional amounts of low-entropy energy (for example, radiant energy from the sun).

In a more general sense, entropy appears to refer to the amount of uncertainty prevailing in a choice situation with many distinguishable alternatives. The amount of entropy displayed by a system is related to the variety, uncertainty or disorder in the system concerned. A reduction of entropy reflects a reduction of the uncertainty in a system: the greater the prior uncertainty of an occurrence, the greater the information gained if such an event occurs. A higher extent of certainty about the order of a system is tied to the number of alternative states of this system, so that the amount of additional information is a measure for the reduction of the entropy of the system. In this sense entropy plays also an important role in information theory (see also Hobson [1971], Jaynes [1957], Mathai and Rathie [1975], Snickers and Weibull [1977], and Theil [1967]), where it is related to the degree of uncertainty

1) The deterministic approach of gravity theory was already criticized by Olsson in 1967.

about the realization of events in information systems.

Entropy is also increasingly being used in social sciences as a descriptive device (hence the name "social physics") related to the state of a social or economic system (for example, in relation to income distribution, budget allocation, input-output linkages etc.). From 1970 there has been a rising tide of applications of entropy theory in the field of spatial interaction models (migration flows, commuting flows, shopping flows, activity allocation, freight flows etc.).

Discussions and applications of entropy theory in relation to spatial interaction modelling can be found among others in Batty [1970], Berry and Schwind [1969], Bussière and Snickars [1970], Cesario [1973], Champenowne et al. [1976], Charnes et al. [1972], Cordey-Hayes and Wilson [1971], Evans [1976], Lowe and Moryadas [1975], Nijkamp [1975] [1976], Nijkamp and Paelinck [1974], Openshaw [1976], Scott [1971], Sheppard [1975], Webber [1976] and Wilson [1970].

Normally, spatial systems are fairly complex and show a high degree of uncertainty. Entropy theory aims to identify the most probable spatial configuration of such a system which may normally adopt an overwhelming number of different states. By means of the entropy concept the most likely equilibrium state of a spatial system may be identified in a manner analogous to physical systems.

Suppose that a spatial system can be characterized by a flow matrix T between all points of origin i and of destination j . Then this matrix T represents the macro-state of this spatial system, which in turn comprises the micro-states (the individual flows t_{ij}). Assuming that the spatial system at hand has a tendency toward maximum entropy, the most likely macro-state is a state with the largest number of ways of occurring of micro-states. The largest number of combinatorial possibilities of pairwise interactions in a spatial system can be represented as:

$$(4.1.) \quad \max v = t! / \prod_{i=1}^I \prod_{j=1}^J t_{ij}!$$

where t is equal to:

$$(4.2.) \quad t = \sum_{i=1}^I \sum_{j=1}^J t_{ij}$$

The maximum value of v represents the maximum number of combinatorial assignments of trips. This number dominates the number of alternative states of the system to such a degree that the spatial allocation of trips associated with this maximum can be regarded as the most likely one. In other words, by specifying a frequency distribution for the number of alternative states one may identify one dominant configuration which overwhelms all the others in

terms of combinatorial possibilities in a probabilistic sense.

Clearly, the disorder of spatial system is restricted by certain constraints. In the case of double constraints (on both origin and destination flows) and of a total travel budget for the whole system, the following constraints have to be added to (4.1.):

$$(4.3.) \quad \sum_{i=1}^I t_{ij} = d_j$$

$$(4.4.) \quad \sum_{j=1}^J t_{ij} = o_i$$

$$(4.5.) \quad \sum_{i=1}^I \sum_{j=1}^J c_{ij} t_{ij} = c$$

where c is the fixed total travel budget of the spatial system¹⁾.

Normally, the t_{ij} 's are unknown, so that the data input consists of the marginal totals of the flow matrix (i.e., o_i and d_j), the unit transportation cost c_{ij} and the travel budget c . The most probable spatial configuration of trips can be determined by calculating the first-order conditions for a maximum value of "objective function" (4.1.) subject to (4.3.) - (4.5.).

Since v is invariant with a monotonically increasing transformation, (4.1.) may be written in logarithmic form as:

$$(4.6.) \quad \max v = \ln t! - \sum_{i=1}^I \sum_{j=1}^J \ln t_{ij}!$$

or following Stirling's approximation:

$$(4.7.) \quad \max v = \ln t! - \sum_{i=1}^I \sum_{j=1}^J (t_{ij} \ln t_{ij} - t_{ij})$$

or:

$$(4.8.) \quad \max v = - \sum_{i=1}^I \sum_{j=1}^J (t_{ij} \ln t_{ij} - t_{ij}) ,$$

where the known constant $\ln t!$ may be left aside.

1) This total travel budget serves to guarantee a certain cost efficiency in the spatial allocation of flows; this budget might also be measured in distance or time dimensions. In a physical sense this constraint would prevent the utilization of unavailable energy.

The Lagrange function for a constrained maximum of v is:

$$(4.9.) \quad L = \ln v + \sum_{i=1}^I \lambda_i (o_i - \sum_{j=1}^J t_{ij}) + \sum_{j=1}^J (d_j - \sum_{i=1}^I t_{ij}) + \beta (c - \sum_{i=1}^I \sum_{j=1}^J c_{ij} t_{ij})$$

The first-order conditions for a constrained maximum yield the following result for the unknown t_{ij} 's:¹⁾

$$(4.10.) \quad t_{ij} = a_i b_j o_i d_j e^{-\beta c_{ij}},$$

where a_i and b_j are equal to:

$$(4.11.) \quad a_i = \left(\sum_{j=1}^J b_j d_j e^{-\beta c_{ij}} \right)^{-1}$$

and:

$$(4.12.) \quad b_j = \left(\sum_{i=1}^I a_i o_i e^{-\beta c_{ij}} \right)^{-1}$$

It is easily seen that (4.10.) is basically the same as (2.7.), because s_{ij} and c_{ij} play a similar role as a distance cost parameter and because the exponential distance friction of (4.10) can be proved to approximate the power distance friction of (2.7.).

The important conclusion is that entropy theory provides a new theoretical background and justification of gravity theory. The expression $e^{-\beta c_{ij}}$ may be regarded as a distance-deterrence function, in which β is related to the aggregate travel costs of the whole spatial system. The greater the value of β , the lower the average travel budget. The parameters a_i and b_j represent the push and pull effects exerted by the points of origin and destination. In formal terms they act as balancing factors such that the additivity conditions are satisfied.

Now that entropy theory appears to provide a new basis for the use of gravity models, one may wonder whether this theory fulfils the methodological requirements mentioned in section 2.

First, the results of the entropy approach guarantee that logical conditions (such as additivity conditions and distance friction conditions) are satisfied. The assumption that each assignment is equally likely provided the constraints

1) The second-order conditions are also satisfied.

are satisfied is, however, less plausible, because precisely here the distance friction is neglected.

Secondly, there is a high degree of similarity between the physical background of entropy and its use in a spatial interaction system. The closeness of the spatial system is guaranteed by conditions (4.3.) and (4.4.), while the notion of fixed mean energy in the system is reflected by the travel budget. The relationship between micro- and macro-states has also a firm analogy to a physical system.

Thirdly, the behavioural background of entropy or its underlying social choice mechanism is not directly clear. There is a priori no reason to believe that people behave like molecules. This problem will be considered in more detail in the next sections ¹⁾.

Next, the mathematical specification of (4.10.) has a rationale in the context of the gravity model, but it may be worth while to investigate whether this specification may be related to a meaningful specification of a utility function in the entropy context (see also sections 5 and 6).

Finally, since (4.10.) is essentially the gravity model, its empirical value is in general satisfactory. Two problems inherent in the use of (4.10.) should be mentioned. First, the assumption of fixed unit transportation cost c_{ij} in a given travel budget c implies that capacity problems of a network are left aside ²⁾. For example, a high flow t_{ij} might exceed the capacity between i and j , so that essentially this congestion phenomenon would lead to an increase of c_{ij} . This possibility is left aside in the entropy approach, although it might be included by means of an iterative adjustment of the travel budget and/or cost coefficient in case of congestion. Secondly, when (4.10.) is used as a projection model for future flows, the calibrated parameters a_i , b_j and β are only valid for a given o_i , d_j , c_{ij} and c . Any future change in these data will affect the parameters a_i , b_j and β , so that (4.10.) cannot be used as a straightforward projection model. Here again an adjustment of parameters has to take place.

The above mentioned remarks hold also true when the doubly ('production-attraction') constrained model is replaced by a singly (either 'production' or 'attraction') constrained model.

- 1) For a discussion on problems of a continuous entropy (in a spatial system) instead of a discrete entropy (in a physical system), see Batty [1974].
- 2) It is clear that abandoning the cost budget would lead to no gravity-type model at all. Furthermore, the capacity problems are also left aside in the traditional gravity approach.

5. Alternative Foundations of the Entropy Model

As set out in the preceding section, the behavioural rationale of entropy theory is still obscure and rather cumbersome. The macro-statistical principle of maximum entropy is not quite clear as a behavioural device. The conflict between social physics and economic utility theory was reflected in studies of Arrowsmith [1973], Beckmann [1974], Beckmann and Golob [1971], Cochrane [1975], and Hansen [1972] [1975] among others. Beckmann and Golob even argued that entropy maximization is a metaphysical approach and that identical results can be achieved by utility maximization. Hansen [1972] proved that the entropy concept is a special case of the utility approach (in Nijkamp [1975] and Nijkamp and Paelinck [1974] it is demonstrated that the entropy approach can be interpreted in terms of a generalized cost function for trip behaviour). Furthermore, Beckmann [1974] constructed an economic model of choice behaviour for trip making that is not only similar to the results of entropy analysis, but that is richer in its interpretation and applicability. Beckmann assumed that a given household is considering various potential residences after having accepted a job in location j . This household is associating a utility index u_i with each residence i . This utility index includes all attractiveness elements of residence i except distance or accessibility. When these utility indices are normally distributed, the probability that a residence has a utility u or more is:

$$(5.1.) \quad p(u) = \int_u^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)/2\sigma^2} dx, \quad ,$$

where μ and σ are the mean and the variance of the quality (acceptability) of housing, respectively. When the household takes into account the distance to work (s_i), then the net utility index of a household living in i is:¹⁾

$$(5.2.) \quad u_i^* = u_i - k s_i, \quad ,$$

where k is conversion factor of distance into utility units. Next, Beckmann assumes that the household is able to specify a minimum achievement level for net utility ($\bar{u}_i$) and that the housing supply in residence i is a_i . Then the total expected number of trips generated by a residential quality at location i at a distance s_i to the work place is:

$$(5.3.) \quad a_i p(\bar{u}_i + k s_i) = a_i e^{(\mu - \bar{u}_i)/\sigma - (k/\sigma) s_i}$$

1) It should be noted that this operation presupposes a cardinal utility concept.

The total work trip distribution from i to j (i.e., t_{ij}) is equal to the number of persons employed in place j (b_j) times the expected residential attractiveness specified in (5.3.):

$$(5.4.) \quad t_{ij} = a_i b_j p(\bar{u}_i + k s_{ij}) \\ \sim a_i b_j e^{-(k/\sigma) s_{ij}}$$

Clearly, (5.4.) is formally analogous to the entropy model (4.10.). Furthermore, this relationship is able to link the achievement level $\bar{u}_i$ and the variance of the utility index of housing to the trip pattern, while in addition to distance also perceived costs play a role. However, there are two problems. The formal derivation of (5.3.) is obscure and should deserve more explanation. Next, (5.4.) only represents the traditional gravity model without the balancing factors from the entropy model.

A more rigorous economic basis for the entropy model along similar lines was provided by Cochrane [1975]. He assumes that trip-makers make decisions which provide the greatest net benefit (utility of any trip after subtraction of trip costs; cf. (5.2.)) for them and that the trip distribution pattern reflects the overall probability of trips being chosen on this basis. Similarly to Beckmann's approach, Cochrane assumes that the probability of any particular possibility offering a utility u within a given range to a trip-maker is given by a single probability density function. Then the probability that a certain trip from place i to j will be made is equal to the probability that a trip from i to j offers a net benefit greater than that of a trip to any other place.

Cochrane postulates next a cumulative distribution function $\varphi(u)$ of u of the greatest among n independent trip samples from a common underlying distribution $f(u)$:

$$(5.5.) \quad \varphi(u) = \{\varphi(u)\}^n$$

Provided n is moderately large, the upper part of $\varphi(u)$ can be shown to be reasonably approximated by means of a simple exponential form:

$$(5.6.) \quad \varphi(u) = \exp \left\{ -n e^{-\lambda(u-\mu)} \right\},$$

where λ is a parameter characterizing the probability density function of u and where μ is the mean. Hence, the probability density function for the utility of the best trip can be determined via the differential of (5.6.).

Next, two additional assumptions are made. First, the number of trip possibilities n_j to place j is a linear function of the number of opportunities d_j at place j :

$$(5.7.) \quad n_j = \theta d_j$$

Secondly, the net benefit u_{ij}^* of a trip from i to j is equal to the gross benefit u_{ij} minus generalized trip costs ks_{ij} minus congestion costs g_j in the place of destination, i.e.,

$$(5.8.) \quad u_{ij}^* = u_{ij} - ks_{ij} - g_j$$

By substituting n_j and u_{ij} into (5.6.) the probability that u^* will attain any particular value is:

$$(5.9.) \quad \varphi_{ij}(u^*) = \exp \left\{ -\theta d_j e^{-\lambda (u^* - \mu + ks_{ij} + g_j)} \right\},$$

where $\varphi_{ij}(u^*)$ is the cumulative distribution function of the net benefit from a preferred trip from i to j . Now the probability p_{ij} that a trip from i to j takes place is equal to the probability that this trip offers a greater net benefit than that to any other zone. This probability can be proved to be:

$$(5.10.) \quad p_{ij} = \frac{d_j e^{-\lambda (ks_{ij} + g_j)}}{\sum_{j=1}^J d_j e^{-\lambda (ks_{ij} + g_j)}}$$

Consequently, the expected number of trips t_{ij} from i to j is:

$$(5.11.) \quad t_{ij} = \frac{o_i d_j e^{-\lambda g_j} e^{-\lambda ks_{ij}}}{\sum_{j=1}^J d_j e^{-\lambda g_j} e^{-\lambda ks_{ij}}}$$

where o_i is the number of trips originating from i . By defining:

$$(5.12.) \quad b_j = e^{-\lambda g_j}$$

the following doubly constrained entropy model is obtained:

$$(5.13.) \quad t_{ij} = \frac{b_j o_i d_j e^{-\lambda ks_{ij}}}{\sum_{j=1}^J b_j d_j e^{-\lambda ks_{ij}}}$$

Cochrane's analysis has demonstrated that a completely different point of departure may lead to the same results as the entropy idea. Therefore, the formal entropy model may also be the result of a utility approach which has a less suspect behavioural foundation than traditional entropy theory. So it appears that the conventional entropy approach may be seriously criticized by behaviouralists who are able to show that a behavioural theory may lead to the same result (see also a recent article by Fisch [1977] who claims that the entropy maximization of expenditure shares is in direct conflict with the standing theory of consumer preferences).

The foregoing remarks suggest that the physical background of the entropy concept is not a prerequisite to arrive at an exponential gravity-type model (see also Smith [1972]), but that also a behavioural utility model may be hypothesized to derive analogous results. Furthermore, it can be proved that information theory, a Bayesian approach via conditional probabilities and maximum likelihood procedures can also be employed to determine the most probable trip distribution pattern in a spatial interaction model. These backgrounds will briefly be discussed now.

In information theory (see Jaynes [1957] and Theil [1967]) an appropriate measure for the information content of a message about the realization of an event with probability p is:

$$(5.14.) \quad m = - \ln p$$

For a set of events i ($i=1, \dots, I$) the mathematical expectation of the information content is:

$$(5.15.) \quad h = - \sum_{i=1}^I p_i \ln p_i \quad ,$$

with:

$$(5.16.) \quad \sum_{i=1}^I p_i = 1 \quad ,$$

so that h reflects the amount of uncertainty represented by a discrete probability distribution.

In the framework of a spatial interaction model where no prior information is available about the trip distribution (except the marginal conditions and the cost budget), it seems to be reasonable to employ a probability distribution with a maximum uncertainty, i.e.,

$$(5.17.) \quad \max v = - \sum_{i=1}^I \sum_{j=1}^J p_{ij} \ln p_{ij} ,$$

where p_{ij} is the trip probability from i to j satisfying the additivity condition:

$$(5.18.) \quad \sum_{i=1}^I \sum_{j=1}^J p_{ij} = 1$$

Furthermore, the marginal conditions and the travel budget may also be introduced:

$$(5.19.) \quad \sum_{j=1}^J p_{ij} = o_i/t$$

$$(5.20.) \quad \sum_{i=1}^I p_{ij} = d_j/t$$

and

$$(5.21.) \quad \sum_{i=1}^I \sum_{j=1}^J p_{ij} c_{ij} = c/t$$

It is clear that this specification gives rise to the same results as the entropy model discussed in section 4. Thus the maximization of the number of micro-states of the system leads to the same results as the maximization of the expected information content of uncertain events. The only difference is that the information approach does not make use of Stirling's approximation formula.

The Bayesian version of the gravity-type model is based on a comparison of any two alternative states of a spatial system $\hat{T}$ and $\hat{T}^*$ with respective trip distribution probabilities $\hat{p}_{ij}$ and $\hat{p}^*_{ij}$. The observed prior probabilities are denoted by π_{ij} . Then the evidence of $\hat{T}$ above $\hat{T}^*$, given the prior state Π , is equal to (see Feller [1968] en Hyman [1969]):

$$(5.22.) \quad e = t \left\{ \sum_{i=1}^I \sum_{j=1}^J \pi_{ij} \ln \hat{p}_{ij} - \sum_{i=1}^I \sum_{j=1}^J \pi_{ij} \ln \hat{p}^*_{ij} \right\}$$

By comparing the state $\hat{T}$ successively with each alternative state, it is clear that the most probable trip distribution pattern is found by maximizing¹⁾

$$(5.23.) \quad e^* = \sum_{i=1}^I \sum_{j=1}^J \pi_{ij} \ln \hat{p}_{ij}$$

Assuming now the general spatial interaction model:

1) It should be noted that this Bayesian criterion function in the entropy context is somewhat different from the usual approach in which the expected posterior loss is minimized, given some loss function.

$$(5.24.) \quad \hat{p}_{ij} = \hat{a}_i \hat{b}_j f(\beta, c_{ij})$$

then the parameters of (5.24.), viz. $\hat{a}_i$, $\hat{b}_j$ and β , have to be calibrated such that (5.23.) is at a maximum. This leads to the following conditions:

$$(5.25.) \quad \sum_{j=1}^J \hat{p}_{ij} = \sum_{j=1}^J \pi_{ij} = o_i/t$$

$$(5.26.) \quad \sum_{i=1}^I \hat{p}_{ij} = \sum_{i=1}^I \pi_{ij} = d_j/t$$

and

$$(5.27.) \quad \sum_{i=1}^I \sum_{j=1}^J \hat{p}_{ij} c_{ij} = \sum_{i=1}^I \sum_{j=1}^J \pi_{ij} c_{ij} = c/t$$

The latter result is entirely analogous to the entropy model if:

$$(5.28.) \quad f(\beta, c_{ij}) = e^{-\beta c_{ij}}$$

and if $\hat{a}_i$ and $\hat{b}_j$ are defined according to (4.11.) and (4.12.).

It is clear that the Bayesian approach to the spatial interaction model leads to the same gravity-type results, but its evident weakness is the prior specification of (5.24.) and (5.28.), so that the exponential and power structure of a gravity model is already imposed a priori.

Finally, the maximum likelihood approach to the spatial interaction model may be mentioned (see also Batty and Mackie [1972], and Evans [1971]). Here again it can be shown that entropy maximization is equivalent to maximizing the likelihood of the spatial macro-state. The production-attraction constrained model is assumed to be:

$$(5.29.) \quad p_{ij} = \bar{a}_i \bar{b}_j e^{-\beta c_{ij}}$$

Assuming that the p_{ij} 's are stochastic and independent variables and assuming a sample of observations on trip flows $\hat{t}_{ij}$ between i and j , the joint probability of the observations $\hat{t}_{ij}$ can be proved to be proportional to:

$$(5.30.) \quad \prod_{i=1}^I \prod_{j=1}^J p_{ij}^{\hat{t}_{ij}} = \prod_{i=1}^I \prod_{j=1}^J \left(\bar{a}_i \bar{b}_j e^{-\beta c_{ij}} \right)^{\hat{t}_{ij}}$$

Therefore, a maximum likelihood estimator of the parameters is obtained by maximizing (5.30) subject to the additivity conditions. Then the parameter

estimators can be proved to be equal to:

$$(5.31.) \quad \bar{a}_i = \pi^{-1} \sum_{j=1}^J \bar{t}_{ij} \left(\sum_{j=1}^J \bar{b}_j e^{-\beta c_{ij}} \right)^{-1}$$

and

$$(5.32.) \quad \bar{b}_j = \pi^{-1} \sum_{i=1}^I \bar{t}_{ij} \left(\sum_{i=1}^I \bar{a}_i e^{-\beta c_{ij}} \right)^{-1}$$

where π is the sample size.

This system can be solved in an iterative way, once β is known. An additional condition from the maximum likelihood procedure appears to be:

$$(5.33.) \quad \pi^{-1} \sum_{i=1}^I \sum_{j=1}^J \bar{t}_{ij} c_{ij} = \sum_{i=1}^I \sum_{j=1}^J p_{ij} c_{ij}$$

On the basis of (5.31) - (5.33.) the unknown parameters (including β) can be calibrated.

It should be noted that the likelihood approach to the spatial interaction model leads to the same expression as the original entropy approach, but here again the multiplicative and exponential spatial interaction model has been assumed a priori.

Our conclusion is that the macro-state probability approach and the information-theoretic approach can be regarded as more adequate foundations for the entropy model than the Bayesian and likelihood approach, because in the former approaches no specific assumptions have been made a priori about the mathematical specification of the spatial interaction model. The Bayesian and likelihood approach are based on a prior specification of a gravity-type interaction model, so that a result which is similar to the entropy result is not a surprise. The last two methods are essentially calibration methods for the spatial interaction model.

Another conclusion is that, in spite of different methodological backgrounds, the gravity-type interaction model appears to be a rather common result. One should be aware, however, that all these different methods are essentially based on similar statistical backgrounds and on similar assumptions (like the fixed travel budget and the absence of explicit behavioural elements).

A final conclusion is that the entropy result can, in principle, also be attained by means of a traditional utility approach, so that it may be

worth while to investigate whether a more general class of utility models may be derived which may be regarded as a welfare basis of the entropy philosophy and which allow us to interpret entropy in behavioural terms. This will be discussed in the next section.

6. A Utility Foundation and Specification of Entropy Theory

In section 5 a survey of alternative justifiable foundations of the entropy model (including a behavioural foundation) has been presented. In the present section an attempt will be made to start off from the entropy philosophy itself and to find a behavioural rationale for entropy maximizing models as such. The essential idea of this approach is that entropy maximizing models can be regarded as a specific type of programming or optimization models. By confronting the general features of programming models with the specific features of entropy maximizing models, a utility background of entropy models may be identified. During the last four years two different lines of thought have been developed in this respect, viz. via the conventional assignment-type models and via the geometric programming models.

The general format of a linear trip assignment model is (see Nijkamp and Paelinck [1976] e.g.):

$$(6.1.) \quad \left\{ \begin{array}{l} \min \varphi = \sum_{i=1}^I \sum_{j=1}^J c_{ij} t_{ij} \\ \sum_{j=1}^J t_{ij} = o_i \quad \forall i \\ \sum_{i=1}^I t_{ij} = d_j \quad \forall j \\ t_{ij} \geq 0 \end{array} \right.$$

Clearly, the structure of such a model bears a certain resemblance to that of an entropy model. Therefore, several researchers have endeavoured to find a link between (6.1.) and (4.1.) - (4.5.) (see among others Coelho and Wilson [1977], Evans [1973], Senior and Wilson [1974], Williams [1976], and Wilson and Senior [1974]).

A closer examination of entropy models and trip assignment models by Evans [1973] learned that the average minimum trip costs of a linear assignment model are equal to the average trip costs of an entropy model, provided the cost friction coefficient approaches to infinity ($\beta \rightarrow \infty$). In this case the optimal trip pattern is equal for both types of models. Should the linear programming model involve multiple solutions, then the entropy outcome can be proved to be one of the solutions. In other words, when

$\beta \rightarrow \infty$ in a maximum entropy model, then the likelihood of observing a minimum travel cost configuration overwhelms the likelihood of any other trip configuration, so that the maximum entropy trip pattern converges in probability to the optimal trip pattern of an assignment model.

In view of these results Senior and Wilson [1974] and Wilson and Senior [1974] claimed that the linear assignment model can be regarded as a specific case of the entropy model, viz. when $\beta \rightarrow \infty$. In our opinion, this statement is somewhat spurious, because the methodological background of linear assignment models and entropy models is entirely different. It is also debatable whether the above mentioned limit case is only valid for entropy models. There is a high degree of evidence that also alternative (and even more general) objective functions, which are optimized subject to a cost constraint, give rise to the same conclusion when $\beta \rightarrow \infty$.

Furthermore, one may question the meaning of $\beta \rightarrow \infty$, especially because the t_{ij} 's and β are inversely related to each other. Assume (4.10.). Then it is easily seen that:

$$(6.2.) \quad \partial t_{ij} / \partial \beta = (A_i + B_j - c_{ij}) t_{ij}$$

with:

$$(6.3.) \quad A_i = a_i^{-1} \partial a_i / \partial \beta$$

and:

$$(6.4.) \quad B_j = b_j^{-1} \partial b_j / \partial \beta$$

It can also be derived by means of (4.3.) and (4.4.) that:

$$(6.5.) \quad \sum_{i=1}^I \partial t_{ij} / \partial \beta = 0$$

and:

$$(6.6.) \quad \sum_{j=1}^J \partial t_{ij} / \partial \beta = 0,$$

so that:

$$(6.7.) \quad \sum_{i=1}^I \sum_{j=1}^J B_j (A_i + B_j - c_{ij}) t_{ij} = 0$$

and:

$$(6.8.) \quad \sum_{i=1}^I \sum_{j=1}^J A_j (A_i + B_j - c_{ij}) t_{ij} = 0$$

The total trip costs of a spatial interaction system are affected by a change in β as:

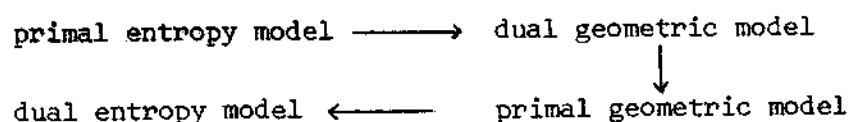
$$(6.9.) \quad \sum_{i=1}^I \sum_{j=1}^J c_{ij} \partial t_{ij} / \partial \beta = \sum_{i=1}^I \sum_{j=1}^J c_{ij} t_{ij} (A_i + B_j - c_{ij})$$

By subtracting (6.7.) and (6.8.) from (6.9.) one obtains:

$$(6.10) \quad \sum_{i=1}^I \sum_{j=1}^J c_{ij} \partial t_{ij} / \partial \beta = - \sum_{i=1}^I \sum_{j=1}^J t_{ij} (A_i + B_j - c_{ij})^2 \leq 0$$

so that there is a monotonically decreasing relationship between total trip costs and β . The assumption of a fixed trip budget made in entropy theory is somewhat contrary to the latter result from traditional spatial interaction theory. As long as the meaning of $\beta \rightarrow \infty$ is obscure (at least in practical model building) and as long as this condition does not affect the total travel cost budget, it is rather difficult to interpret linear programming models as a special type of entropy models. Therefore, instead of interpreting entropy theory in behavioural terms by means of the limit case of a linear assignment model, it may be more fruitful to find a behavioural interpretation for the entropy model itself.

The alternative way to find a behavioural rationale for entropy maximizing models is the use of geometric programming theory. This approach is not based on a limit case of entropy models, but on a dual specification and interpretation of entropy maximizing models. A first contribution to a utility interpretation of entropy models via geometric programming theory was offered by Nijkamp and Paelinck [1974], while subsequent contributions were offered by Charnes et al. [1976], Dinkel et al. [1977], Kadás and Klafszky [1977], Nijkamp [1975] [1976], and Scott and Jefferson [1977]. The approach presented here is based on a new (unconstrained) geometric interpretation. The essential idea of this approach is that entropy maximizing models should have a dual which may shed more light on the shadow values (and hence on the economic valuation) of the primal entropy model. It can be shown that the specification of a entropy maximizing model is a specific case of a dual geometric programming model. Therefore, the primal version of a geometric programming model can be used to derive a dual version of the entropy model:



For the sake of simplicity the entropy maximizing model will be presented here in terms of relative trip proportions p_{ij} between i and j :

$$(6.11.) \quad p_{ij} = t_{ij}/t$$

In other words, the entropy model (5.17.) - (5.21.) forms the basis of the analysis. Now it is easily seen that this model is equivalent to:

$$(6.12.) \quad \left\{ \begin{array}{l} \max v = - \sum_{i=1}^I \sum_{j=1}^J p_{ij} \ln p_{ij} \\ \sum_{i=1}^I \sum_{j=1}^J p_{ij} = 1 \\ - \frac{o_i}{t} \sum_{j=1}^J p_{ij} + \sum_{j=1}^J p_{ij} = 0 \quad v_i \\ - \frac{d_j}{t} \sum_{i=1}^I p_{ij} + \sum_{i=1}^I p_{ij} = 0 \quad v_j \\ - \sum_{i=1}^I \sum_{j=1}^J c_{ij} p_{ij} + \frac{c}{t} \sum_{i=1}^I \sum_{j=1}^J p_{ij} = 0 \\ p_{ij} \geq 0 \end{array} \right.$$

For a situation of 2 places of origin and 2 places of destination the foregoing model can be written as:

$$(6.13.) \quad \left\{ \begin{array}{l} \max v = - p_{11} \ln p_{11} - p_{12} \ln p_{12} - p_{21} \ln p_{21} - p_{22} \ln p_{22} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1-o_1/t & 1-o_1/t & -o_1/t & -o_1/t \\ -o_2/t & -o_2/t & 1-o_2/t & 1-o_2/t \\ 1-d_1/t & -d_1/t & 1-d_1/t & -d_1/t \\ -d_2/t & 1-d_2/t & -d_2/t & 1-d_2/t \\ c/t-c_{11} & c/t-c_{12} & c/t-c_{21} & c/t-c_{22} \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \right.$$

The total number of unknown variables of model (6.12.) is IJ . Now it is easily verified that the latter model is a specific member of the class of dual geometric programming models¹⁾. Therefore, it may be worth while

1) See for a general discussion of geometric programming models Duffin et al. [1967] and Nijkamp [1972].

to derive its corresponding primal version. The primal geometric programming model associated with (6.12.) can adopt several forms (see Nijkamp [1975], e.g.). A rather compact form can easily be derived via the primal-dual geometric relationships as:

$$(6.14.) \quad \min \Psi = \left(\begin{array}{ccc} I & J & o_i/t \quad d_j/t \quad -c/t \\ \Pi & \Pi & x_i \quad y_j \quad z \end{array} \right)_{i=1 \quad j=1}^{-1} \left(\begin{array}{ccc} I & J & \\ \Sigma & \Sigma & x_i \quad y_j \quad z \end{array} \right)_{i=1 \quad j=1}^{-c_{ij}}$$

This primal geometric programming model is an unconstrained model. It can be regarded as the dual specification of the conventional entropy model. The variables x_i, y_j and z are dual variables ('shadow prices') related to the origin constraints, the destination constraints and the travel budget, respectively. There is a one-to-one relationship between the primal and dual variables of a geometric program:

$$(6.15.) \quad p_{ij} = \Psi^{-1} \left(\begin{array}{ccc} I & J & o_i/t \quad d_j/t \quad -c/t \\ \Pi & \Pi & x_i \quad y_j \quad z \end{array} \right)_{i=1 \quad j=1}^{-1} \left(x_i \quad y_j \quad z \right)^{-c_{ij}},$$

so that p_{ij} can be interpreted as the relative proportion of the term (i,j) from the primal geometric program. The relationship between the primal and dual objective function at the optimum can be represented by means of the following one-to-one primal-dual condition:

$$(6.16.) \quad v = \ln \Psi$$

Given (6.15.) and (6.16.), there is a unique relationship between a primal and a dual geometric programming model. Consequently, both models can be transformed into each other.

Now the question has to be answered: how to interpret the primal geometric model (i.e., the dual entropy model)? The shadow prices of the primal entropy model are x_i (associated with o_i), y_j (associated with d_j) and z (associated with c). The dual entropy model is composed of two parts X and Y , defined as:

$$(6.17.) \quad X = \begin{array}{ccc} I & J & \\ \Sigma & \Sigma & -o_i/t \quad -d_j/t \quad c/t \\ i=1 \quad j=1 & x_i & y_j \quad z \end{array}$$

and:

$$(6.18.) \quad Y = \begin{array}{ccc} I & J & \\ \Sigma & \Sigma & x_i \quad y_j \quad z \end{array}^{-c_{ij}}$$

The exponents o_i/t , d_j/t and c/t in (6.17.) represent the relative influence of point i , the relative influence of point j and the average travel

budget, respectively. These exponents can be regarded as the weights in the geometric average of all x_i 's, y_j 's and z . The shadow cost variables x_i , y_j and z are related to the imputed marginal potential spatial interaction (measured in entropy terms) caused by the presence of the origin constraints, the destination constraints and the travel budget respectively. Therefore, $x_i^{-o_i/t}$, $y_j^{-d_j/t}$ and $z^{c/t}$ may be regarded as weighted shadow variables related to push effects, pull effects and the total travel budget. X is essentially a formal expression for a generalized dual gravity model in which all points of origin and of destination as well as the average trip budget are included, so that X represents the shadow costs of the weighted aggregate interaction of the total spatial system based on attraction and repulsion effects of all places and controlled by the total travel budget. This indicates that an increase in the travel budget c will lead to a higher aggregate interaction, whereas an increase in an individual o_i or d_j may exert a negative influence. In fact, $z^{c/t}$ reflects a travel cost opportunity factor, as is easily seen by writing it as:

$$(6.19.) \quad z^{c/t} = e^{(c/t) \ln z},$$

which indicates that $\ln z$ is essentially a cost opportunity coefficient.

The second term Y has also some interesting features. The individual terms of Y , defined as:

$$(6.20.) \quad y_{ij} = x_i y_j z^{-c_{ij}}$$

are related to the pairwise interactions between all points of origin and of destination. Here again the variables x_i , y_j and z are related to the push power of a point of origin, the pull power of a point of destination, and to the cost friction of the travel budget, respectively. y_{ij} may also be written as:

$$(6.21.) \quad y_{ij} = x_i y_j e^{-c_{ij} \ln z},$$

which bears a close resemblance to the original gravity model, so that $\ln z$ can again be regarded as a shadow cost parameter for bridging the interzonal distances. Obviously, the pairwise interaction between i and j declines as the unit trip costs rise. Therefore, it is conceivable to interpret (6.21.) as the imputed marginal costs of interaction caused by a trip from i to j , given the trip costs c_{ij} . Hence, the term Y is associated with the total shadow costs of pairwise spatial interaction between all points of origin

and of destination.

Minimization of (6.14.) implies that the pairwise shadow push and pull costs (based on interzonal cost friction) multiplied by the aggregate shadow costs of interaction arising from the total travel budget are at a minimum. The meaning of this statement can be clarified by using (6.15.) to find an expression for the ratio of (relative) trips p_{ij} and p_{ik} :

$$\begin{aligned} (6.22.) \quad p_{ij} / p_{ik} &= (y_j / y_k) z^{-(c_{ij} - c_{ik})} \\ &= (y_j / y_k) e^{-(c_{ij} - c_{ik}) \ln z} \end{aligned}$$

According to (4.10.) the ratio of relative trips calculated by means of the entropy model is:

$$(6.23.) \quad p_{ij} / p_{ik} = (b_j d_j / b_k d_k) e^{-(c_{ij} - c_{ik}) \beta}$$

The latter result bears a close resemblance to (6.22.), so that y_i is the dual expression for the pull effect of j , and $\ln z$ indeed a dual expression for the traditional cost friction coefficient β .

The foregoing analysis gives rise to two important conclusions. First, by means of geometric programming theory the dual entropy relationships can be derived. This appears to allow a more behavioural (or at least economic) interpretation in terms of shadow costs of spatial interactions. Clearly, due to the non-linearity this economic background is less straightforward than in usual transportation models, but a close connection with traditional spatial interaction theory could be demonstrated, so that entropy theory appears to have a macro-behavioural rationale.

Secondly, by writing entropy models as geometric programming models the traditional entropy can easily be extended with additional constraints (for example, capacity constraints). In the traditional entropy approach it is extremely difficult to add new constraints, because this will normally preclude an analytical expression for the equilibrium trip pattern. By including these constraints in a geometric programming model, one may use the solution algorithms for geometric programming models to calculate in an efficient manner the optimal (equilibrium) flows especially in those cases, for which the original entropy solution does no longer hold. But also in the normal entropy model, the geometric programming approach may be used to calculate the optimal trip pattern (see for an application also Kádas and Klafszky [1976]).

Next, it may be worth while to investigate whether some additional properties of (6.14.) can be derived. First, one may analyse the marginal effect of a change in x_i upon Ψ :

$$(6.24.) \quad \frac{\partial \Psi}{\partial x_i} = -x_i^{-1} (o_i/t) XY + x_i^{-1} X \sum_{j=1}^J Y_{ij} \\ = \left\{ -(o_i/t) Y + \sum_{j=1}^J Y_{ij} \right\} x_i^{-1} X$$

Therefore, $\frac{\partial \Psi}{\partial x_i} > 0$, if the aggregate shadow value of spatial interaction from place i onward (i.e., $\sum_{j=1}^J Y_{ij}$) exceeds the weighted shadow costs of the total spatial interaction originating from i (i.e., $(o_i/t) Y$).

A similar result can be achieved for the shadow pull variable y_i . The marginal effect of z can be calculated as:

$$(6.25.) \quad \frac{\partial \Psi}{\partial z} = z^{-1} (c/t) XY - z^{-1} X \sum_{i=1}^I \sum_{j=1}^J c_{ij} Y_{ij}$$

It is easily seen that the marginal effect of z is positive, if the average travel budget (c/t) is higher than the aggregate costs of imputed spatial interaction $\left(\sum_{i=1}^I \sum_{j=1}^J c_{ij} Y_{ij} \right)$.

Another property may be derived by using some results from vector optimization theory¹⁾. The individual terms Y_{ij} from (6.14.) can be regarded as individual shadow cost objectives, which are included in Y in an unweighted form. It is a well known property of geometric-arithmetic averages (cf. Pascual and Ben-Israel 1971) that:

$$(6.26.) \quad \sum_{i=1}^I \sum_{j=1}^J Y_{ij} \geq IJ \prod_{i=1}^I \prod_{j=1}^J \left(Y_{ij} \right)^{\frac{1}{IJ}}$$

so that:

$$(6.27.) \quad \Psi \geq IJ X \prod_{i=1}^I \prod_{j=1}^J \left(Y_{ij} \right)^{\frac{1}{IJ}} \\ = IJ \prod_{i=1}^I \prod_{j=1}^J x_i^{o_i/t + 1/I} y_j^{-d_j/t + 1/J} z^{c/t - \sum_{i=1}^I \sum_{j=1}^J c_{ij}/IJ}$$

It should be noted that the exponents in (6.26.) compare the trip propor-

1) See for a survey and applications Nijkamp [1977] and Nijkamp and Rietveld [1976].

tions with the average unweighted shares, so that the direction of the marginal effects of x_i , y_j and z is determined by their successive mutual orders of magnitude. It is easily seen that (6.26.) is an easier minimizing program than (6.14.), so that it might be used to calculate a minimum level for the entropy expression.

7. Extensions and Applications of Entropy Models.

The entropy models discussed in the foregoing sections can be extended in several ways. One possible approach is to introduce the notion of a continuous entropy which is related to continuous situations in which no discrete spatial points are distinguished (for example, a distribution of population through a continuous space). Continuous entropy models were studied among others by Batty [1974], Casario and Zerdy [1975], Hobson and Cheng [1973], Ingels [1971], and Jaynes [1973].

Another extension of entropy theory is to introduce the notion of a conditional entropy. Assume a trip probability p_{ij} . Then a conditional probability is defined as the probability that a trip will terminate in place j , given the condition that this trip originates from place i (cf. Theil [1967]). The entropy function associated with such a conditional probability is:

$$(7.1.) \quad w = - \sum_{i=1}^I \sum_{j=1}^J (p_{ij}/p_{i.}) \ln (p_{ij}/p_{i.}) ,$$

where the marginal probability $p_{i.}$ is defined as:

$$(7.2.) \quad p_{i.} = \sum_{j=1}^J p_{ij}$$

Next, the expected (average) conditional entropy of the trips attracted by all zones of destination, given the known prior probabilities for the total trips from all points of origin, are:

$$\begin{aligned} (7.3.) \quad w^* &= - \sum_{i=1}^I p_{i.} \sum_{j=1}^J (p_{ij}/p_{i.}) \ln (p_{ij}/p_{i.}) \\ &= - \sum_{i=1}^I \sum_{j=1}^J p_{ij} \ln (p_{ij}/p_{i.}) \end{aligned}$$

It is easily seen that the latter format corresponds again to the objective function of a dual geometric model, so that problems related to (7.3.) can be solved by means of standard algorithms for geometric programming problems.

The foregoing approach can be generalized to any other prior distribution (see among others Batty and March [1976], Bussière and Snickars [1970], Cesario and Zerdy [1975], Hobson [1971], Hobson and Cheng [1973], Good [1963], Jaynes [1968], Kádas and Klafszky [1970], Kullback [1959], Lindley [1972], Nijkamp [1976] and Snickars and Weibull [1977]). This implies that p_{ij} may be replaced by any prior probability on certain flows which may be known, so that this prior probability is a first guess of trip probabilities (a hypothetical estimation) on the basis of whatever information is available, prior to the introduction of new information from the spatial interaction model. In this respect, Hobson has shown that:

$$(7.4.) \quad w^0 = \sum_{i=1}^I \sum_{j=1}^J p_{ij} \ln (p_{ij} / \hat{p}_{ij})$$

is a unique measure of the information content (in entropy terms) in a posterior probability assignment p_{ij} , given a prior probability $\hat{p}_{ij}$.

An application of the latter approach is contained in Nijkamp [1976], in which commuting flows were assessed by means of a prior probability entropy model. The prior probabilities of commuting flows were derived from a partial behavioural trip model, in which interlocal trips were explained on the basis of environmental and residential characteristics in the places of origin and employment characteristics in the places of destination. By introducing these prior probabilities in the posterior entropy model, the entropy model itself can be used as a consistent spatial interaction framework to assess the posterior trip probabilities.

This integration of a behavioural model and an entropy model can be regarded as an extremely important means to solve the behavioural-physical dilemma in social physics. Other applications of this approach can be found among others in Batty and March [1976], Kádas and Klafszky [1976] and Snickers and Weibull [1977].

During recent years a wide variety of entropy models has been applied in many fields: budget allocation models, consumer expenditure models, trip assignment models, input-output models, migration and commuting models, transportation models, location models and so forth. In Nijkamp [1975] it was demonstrated that entropy results differ significantly from linear programming results. An obvious difference is already that in a linear programming framework the number of non-zero variables does not exceed the total number of independent side-conditions. Next, the entropy results appear to reflect a much higher spatial dispersion of flows in a spatial interaction model than linear programming results. Furthermore, cross-hauling may take place in entropy models, whereas this is formally impossible

in linear programming models.¹⁾ Finally, a comparison of actual flows with entropy and linear programming results learns that, in general, entropy models tend to underestimate intrazonal flows, whereas linear programming models tend to overestimate intrazonal flows. The latter phenomenon is the result of the entropy philosophy in which the number of micro-states in a certain macro-state is maximized.

Further discussions on the empirical validity of entropy models can be found among others in Batty and Mackie [1972], Cripps and Cater [1971], Hathaway [1975], Kirby [1974], Openshaw [1975], and Senior and Wilson [1974]. In these studies the calibration problems of entropy models are also set out in more detail.

8. Conclusion and Evaluation.

The idea of entropy has opened a rich field of scientific research, in spite of the fact that the meaning of entropy varies substantially in the field of social sciences. First, entropy can be regarded as a background idea for studying the stability and equilibrium of a system (see the excellent study of Georgescu - Roegen [1971] on the ecological basis and constraints of economic systems). Secondly, entropy can be employed as an operational tool for solving allocation and assignment problems.

With respect to the latter issue, entropy constitutes the foundation for the traditional gravity model. For both gravity and entropy models a set of criteria have been specified in order to judge the plausibility of these methods in social sciences. Both methods appeared to have a behavioural economic-oriented interpretation. Therefore, the conclusion is that from a methodological point of view the use of gravity and entropy models is logically sound and plausible. The empirical results support once more the use of gravity and entropy models.

On the other hand, gravity and entropy theory as a tool in social physics does not provide a neutral device for solving allocation and assignment problems. Both theories are related to a rather specific utility background as is shown by the use of geometric programming theory. Therefore, it is reasonable to conclude that gravity and entropy theory are only one class of spatial interaction theories. Entropy models may be extended with behavioural relationships in order to overcome the restrictive assumptions

1) It is clear that in reality cross-hauling is more probable, as the homogeneity of the spatial variables (commodities, e.g.) is lower.

of these models (the fixed travel budget, e.g.). This will directly lead to the use of geometric programming methods. A complementary approach may be to include the results of a behavioural approach as a prior trip probability in a Bayesian type of entropy model. Therefore, this paper can be regarded as a plea for a better integration of behavioural models and models from social physics. In this respect, the present author agrees with the statement made by Beckmann [1974]: "A good utilitist will not be outdone by an entropist".

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