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Abstract

In this paper, we address the problem of maximizing expected return subject to a worst case return constraint by composing a portfolio that may consist of cash, holdings in a stock market index and options on the index. We derive properties of optimal and feasible portfolios and present a linear programming model to solve the problem. The optimal portfolios have pay-off functions that reflect a gambling policy. We show that optimal solutions to a large class of portfolio models that maximize expected return subject to downside risk constraints are driven by this casino effect and present tractable conditions under which it occurs in our model. We propose to control the casino effect by using chance constraints. Using results from financial theory we formulate an LP model that maximizes expected return subject to worst case return constraints and chance constraints on achieving prespecified levels of return. The results are illustrated with real life data on the S&P 500 index.

Introduction

In the asset management industry, the concept of guaranteed return products seems to become more and more important. As at June 1996, the market value of mutual funds with a guaranteed minimum level of return, listed at European exchanges, was well beyond 120 billion Deutschmark. Institutional investors and corporate treasuries also show a rapidly growing interest in asset management styles with absolute return guarantees. In many cases this interest is sparked by the need to have exposure to risky asset classes with high expected returns while not being able to bear the risk of substantial losses in the short run. Even though specific liability structures may call for well tuned investment policies, many professional investors have a generic investment problem in common: to maximize expected return while limiting downside risk. Traditionally this trade-off was reflected by the choice of asset mix. Nowadays, better trade-offs can be achieved by the use of derivative financial instruments such as options.

In this paper, we suggest generalizations and improvements of Dert and Oldenkamp (1996) and focus on the problem of determining an investment portfolio from a universe of assets that consists of cash, a stock index and European, exchange listed options on the index that expire at the investment horizon, such that:

- the ***expected*** return at the investment horizon is maximized, subject to
- the *realized* portfolio return at the horizon is no worse than the ***guaranteed*** return, independent of the value of the index, and
- the probability that the portfolio return will exceed a given target return is sufficiently large.

It is our experience that investors are very able to state their investment preferences in terms of the framework of the above problem description. Although more complex utility functions could also serve to reflect risk-return trade-offs, we feel that such an approach would relate less well to the frame of reference of many investors in practice.

Whereas there is abundant literature on the trade-off between expected return and standard deviation of return, initiated by Markowitz (1952), and the trade-off between expected return and downside risk (Leibowitz and Henriksson (1989), Harlow (1991), Leibowitz and Kogelman (1991), Pellsser and Vorst (1995)), little has been published on the trade-off between expected return and the level of guaranteed return, using derivative instruments. Pelsser and Vorst (1995) concentrate on expected return maximization subject to shortfall risk constraints, by composing portfolios consisting of stocks and European stock options. Their approach is based on the assumption that the continuous version of the CAPM holds.

The remainder of this paper is organized as follows. First we discuss the problem without chance constraints and derive necessary and sufficient conditions for feasibility and optimality of solutions to this problem. The conditions can be formulated in such a way that it can easily be verified whether they are met for given portfolios. They are derived without making assumptions on the process that generates returns on the stock index. Using these conditions, we present an LP model to determine optimal portfolios, based on actual market prices, taking bid-offer spreads into account explicitly.

In section 3, we discuss the casino effect, which can be shown to be a property of optimal solutions to a large class of optimization models of which the model that we develop in this paper can be seen as a special case. Many investors may perceive the casino effect as an undesirable property of the pay-off functions of their portfolio. Therefore, we extend our model with chance constraints that can be used to control the casino effect. To formulate the extended problem as an LP model, we use a result from financial theory (Dybvig (1988)) that has been derived under the assumption of lognormally distributed returns on the index.

The theoretical results are illustrated by numerical examples that use real-life market data of options on the S&P 500 index, one of the leading stock market indices in the USA.

1 Notation and Assumptions

Before we discuss our model and solution procedure in more detail, we introduce some notation.

- T: the investment horizon,
- $S(t), t \in [0, T]$: the value of the index at time t (S refers to the index value at the horizon T, $S(T)$, and F to its distribution function),
- n: the number of different exercise prices of options in our model,
- $K_i, i = 1, 2, \ldots, n$: the exercise price of the i-th option, $K_1 < \ldots < K_n$,
- $p_i, c_i, i = 1, 2, \ldots, n$: the prices of European put and call options with exercise prices K_i, respectively, expiring at the horizon T,
- $\alpha_i^p, \alpha_i^c, i = 1, 2, \ldots, n$: the expected pay-offs of European put and call options with exercise prices K_i, respectively, expiring at the horizon T,
- $(d)^+$: the positive part of d, equalling $\max\{d, 0\}$,
- $x_i^p, x_i^c, i = 1, 2, \ldots, n$: the amount of put and call options in the portfolio,
- y: the number of units of the index in the portfolio,
- z: the amount of money invested in the risk-free asset,
- r: the risk-free rate with maturity T,
- $V(S; x^p, x^c, y, z)$: the pay-off or value of the portfolio consisting of y units of the index, an investment in the risk-free asset of z and a portfolio (x^p, x^c) of index options at the horizon, as a function of the of the index level at the horizon, S,
- θ: the pay-off level to be guaranteed at the horizon.

Furthermore, we will make the following assumptions:

- The index used in the model is a total return index, all dividends are immediately reinvested.
- We consider a single period problem, i.e. once the option portfolio is bought, it will be held until the horizon.
- Instead of using returns, we will scale the initial value of the index to one dollar (so S(0) = 1) and make use of portfolio pay-offs.
- The initial budget is equal to one dollar, so the amounts invested in the index, index options and the risk-free asset sum up to one.
- There are no execution costs, other than the bid-offer spread.

2 Guaranteed and Expected Returns: Theory

In this section, we will analyze the trade-off between the guaranteed and expected pay-off in theory. It should be noted that the optimization approach presented in the following sections presumes the use of real-life options, with prices possibly deviating from their theoretical Black-Scholes values. In the presence of theoretical option pricing, financially engineered guaranteed return products, like PEN's (Protected Equity Notes, which consist of a zero-coupon bond and at the money call options), may be constructed. They have already been analyzed in some detail in the literature (Merrill and Thorley (1996)).

In this section, we first introduce a general problem formulation. We subsequently derive necessary and sufficient conditions for feasibility and optimality of a given portfolio. Based on these conditions, we present a linear programming model to determine optimal portfolios of zero-coupon bonds, a position in the underlying index and index options. Section 2.4 presents a numerical illustration based on real-life data of the S&P 500 index.

2.1 A Conceptual Problem Formulation

Given the assumptions stated in section 1, we derive the following expression for the pay-off at the horizon:

$$V(S; x^p, x^c, y, z) = yS + ze^{rT} + \sum_{i=1}^{n} \{ x_i^p (K_i - S)^+ + x_i^c (S - K_i)^+ \}, \tag{1}$$

Notice that the pay-off function is linear in the portfolio holdings x^p, x^c, y and z. The objective is to allocate a budget to the amount of 1 over a universe of assets so as to maximize the expected pay-off of the portfolio, subject to the pay-off at the horizon, $V(S; x^p, x^c, y, z)$, always being larger or equal to the guaranteed pay-off level. This is reflected by the following stochastic programming problem:

$$
(P) \qquad \max_{x^p, x^c, y, z} \quad e^{\mu T} y + e^{rT} z + \alpha^{p\prime} x^p + \alpha^{c\prime} x^c \tag{2}
$$

$$
p' x^p + c' x^c + y + z \leq 1 \quad , \tag{3}
$$

$$
V(S; x^p, x^c, y, \ z) \ \geq \ \theta, \qquad \forall S \geq 0 \tag{4}
$$

$$
x^p, \, x^c \in \mathrm{R}^n, \, y, \, z \in \mathrm{R}.
$$

For notational convenience, we have ignored bid-offer spreads in this formulation. They can easily be incorporated in this model by introducing separate variables for short and long positions of options. The numerical results in 2.4 and 3.2 have been obtained by a model that does take bid-offer spreads into account. The stochastic nature of this problem is caused by the dependence of the pay-off on the uncertain value S of the index at the horizon. From a computational point of view, this model is rather unattractive because constraint (4) is one with infinite dimension. In section 2.2, a finite set of constraints will be derived that can be used instead of constraint (4).

2.2 Conditions for Feasibility

By assumption, all options expire at the horizon. It follows that the pay-off function of each of the option series is a piecewise linear function of the index value at the horizon. As a consequence, the pay-off function of any portfolio of such options is also piecewise linear in the index value at the horizon. It is easy to verify that the breakpoints of the pay-off function coincide with the excercise prices of the options in the portfolio. These observations enable one to replace the infinitely dimensional constraint (4) by $n + 2$ linear constraints without affecting the feasible region. For $K_1 \leq S \leq K_n$, it suffices to require the pay-off at all excercise prices to be greater than or equal to the guaranteed level of pay-off. For $S \geq K_n$ and for $S \leq K_1$, the pay-off function is linear in S. Thus, it takes only two additional constraints to ensure a sufficient level of pay-off outside of the interval $[K_1, \, K_n]$ as well. This insight is formalized in the following lemma.

Lemma 2.1 *Let $(x^p, \, x^c, y, \ z)$ be an optioned portfolio, satisfying the budget constraint (3). Then, the portfolio is feasible to problem (P) if and only if*

$$
V(0; x^p, x^c, y, z) \ \geq \ \theta, \tag{5}
$$

$$
V(K_i; x^p, x^c, y, z) \ \geq \ \theta, \qquad i = 1, 2, \ldots, n, \tag{6}
$$

$$
\sum_{i=1}^{n} x_i^c + y \ \geq \ 0. \tag{7}
$$

PROOF: We begin by taking the derivative of the pay-off function V with respect to S in an arbitrary point $S \neq K_i$, $i = 1, 2, \ldots, n$:

$$
\begin{aligned}
V'(S) &= y - \sum_{i=1}^{n} x_i^p, && 0 < S < K_1, \\
&= y - \sum_{i=j}^{n} x_i^p + \sum_{i=1}^{j-1} y_i^c, && K_{j-1} < S < K_j, \\
&= y + \sum_{i=1}^{n} x_i^c, && K_n < S.
\end{aligned}
$$

Note that this derivative is a constant between breakpoints, which implies that the pay-off function is linear in between the breakpoints K_i. As can easily be verified, V is continuous in these breakpoints. Hence, if the guaranteed pay-off constraints are satisfied in $S = 0$ and in each breakpoint, then the minimum pay-off requirement is also met $\forall S \in [0, K_n]$, since the pay-off in this interval is always a convex combination of the pay-off of two points in the set $0, K_1, \ldots, K_n$. Since V is linear in S for $S \geq K_n$, it is necessary and sufficient to require the derivative of V to be nonnegative for $S \geq K_n$, given $V(K_n; x^p, x^c, y, z) \geq \theta$. ❑

Table 1:
Data for S&P 500 options (may 1997)

	bid	*mid*	*ask*	*implied*	*expected*	*return (ask)*	*return (bid)*
S&P 500	760.48	760.48	760.48		765.32		
call 740	27.63	28.12	28.52	19.4	28.24	.9868	1.0222
call 750	20.50	21.00	21.50	18.7	21.37	.9937	1.0423
call 760	14.25	14.62	15.00	17.7	15.57	1.0383	1.0930
call 770	9.50	9.75	10.00	17.2	10.90	1.0897	1.1471
call 780	5.63	5.88	6.13	16.5	7.31	1.1925	1.2984
call 790	3.00	3.19	3.38	15.8	4.69	1.3883	1.5642
call 800	1.50	1.69	1.88	15.6	2.88	1.5305	1.9282
put 740	6.50	6.69	6.88	19.6	5.34	.7760	.8214
put 750	9.25	9.50	9.75	18.8	8.46	.8681	.9150
put 760	13.25	13.42	13.62	18.2	12.67	.9303	.9563
put 770	18.00	18.38	18.75	17.5	17.99	.9597	.9996
put 780	23.75	24.25	24.75	16.6	24.41	.9861	1.0276
put 790	31.25	31.75	32.25	16.3	31.79	.9857	1.0172
put 800	39.63	40.13	40.63	16.1	39.97	.9831	1.0087

2.3 A Tractable Model

Replacing inequality (4) by (5), (6) and (7), we obtain the following linear programming (LP) formulation for problem (P).

$$(P') \qquad \max_{x^p, x^c, y, z} e^{\mu T} y + e^{rT} z + \alpha^{p\prime} x^p + \alpha^{c\prime} x^c \tag{8}$$

$$p' x^p + c' x^c + y + z \leq 1, \tag{9}$$

$$V(0; x^p, x^c, y, z) \geq \theta, \tag{10}$$

$$V(K_i; x^p, x^c, y, z) \geq \theta, \quad i = 1, 2, \ldots, n, \tag{11}$$

$$y + \iota' x^c \geq 0, \tag{12}$$

$$x^p, x^c \in \mathrm{R}^n, y, z \in \mathrm{R}.$$

An LP formulation in the variables x^p, x^c, y and z can easily be obtained now by substituting the explicit expression (1) for the pay-off function V into the problem above. This LP problem has $2n+2$ free variables and $n+3$ inequality constraints. It can routinely be solved by standard LP solvers. In section 2.4, we illustrate this approach by solving a specific instance of the problem defined by market data of the Chicago Board Options Exchange.

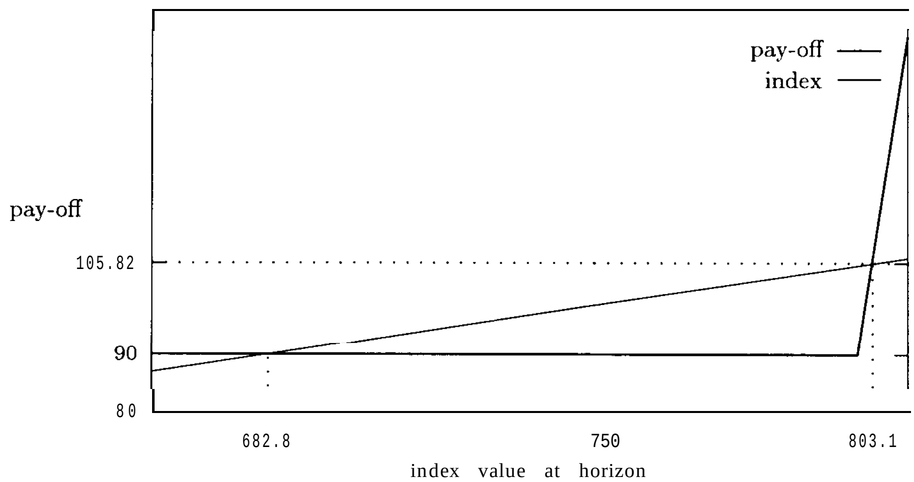

Figure 1: The optimal pay-off function

2.4 An Example With the S&P 500 Index

We will report results based on market data of options on the S&P 500 index, that are listed on the Chicago Board Options Exchange (CBOE). The prices are drawn from a Reuters screen in the beginning of the afternoon of April 22, 1997. Table 1 contains the input to the LP model. The horizon is equal to the expiration date of the options, which is May 15, 1997. The

Table 2:
Optimal solution for various guaranteed levels

θ	cash	800 call	*exp.* ret.
.90	89.67	5.495	5.81
.91	90.67	4.965	5.29
.92	91.66	4.435	4.76
.93	92.66	3.905	4.24
.94	93.66	3.375	3.71
.95	94.65	2.845	3.19
.96	95.65	2.315	2.66
.97	96.64	1.785	2.14
.98	97.64	1.255	1.61
.99	98.64	.725	1.09
1.00	99.63	.195	.56

investment horizon is 23 days. The value of the index at the beginning of this period was 760.48. The present value of estimated dividends is 1.59 dollar.
The starting portfolio consists of 100 dollars cash. The expected pay-offs of the options have been calculated under the assumption that the index value at the horizon is lognormally distributed with an expected annualized growth rate of 10% (excluding dividends, this corresponds to an instantaneous growth rate of 7.9%) and an annualized standard deviation of 17.9%, which is in conformity with the implied volatility of the at the money options in the universe (see table 1). Note from table 1 that the expected pay-offs (in pay-off column) for the puts are always lower than the premia, whereas for the calls, the opposite is true. This reflects the assumption that there is a positive risk premium for the index in our model. We have assumed that one can lend and borrow against an annualized short term interest rate of 5.51%, which corresponds to a return at the investment horizon of 0.37%.
The optimal solutions for different values of the minimally required return are given in tabel 2. It is easy to interpret the solutions: for example: in order to guarantee a pay-off of 90, 89.67 is invested in the risk-free asset in order to secure the minimally required pay-off of 90. The remainder of budget can now be invested in the portfolio with the highest expected return, provided that it can never generate a negative pay-off. In this example, this portfolio consists of the call 800 only.
The optimal pay-off function corresponding to this solution is given in figure 1 (the straight line corresponds to the pay-off of the underlying index). Since the optimal portfolio contains a substantial position in out of the money calls, large gains will be realized if the index rises by more than 5.2% (i.e. higher than 800). For index returns up to 5.2%, however, the return on the portfolio is always equal to the minimally required return of -10%. This portfolio reflects a gambling policy: receiving a very high return with small probability, and a low return with

high probability. This phenomenon will be referred to as the casino *effect.* For a more formal definition of the casino effect, the reader is referred to definition A.1 in the appendix, which also contains the derivation of necessary conditions for the casino effect.

For different levels of the guarantee θ, the solutions look strikingly similar. The optimal option portfolio always involves a long position in the risk-free asset, equal to the amount of the present value of the minimally required pay-off, together with a long position in the calls 800. The size of this latter position decreases linearly with the guaranteed pay-off level.

3 The Casino Effect

Table 3:
Optimal solutions with guarantee of -10%, chance constraint at 0% level

	unrestricted	*chance*	*constraint*
		50%	70%
exp. return	5.81 %	1.77 %	0.63 %
cash	89.67	99.63	89.65
index			0.013
call 740			-0.013
call 800	5.495	1.889	0.238
put 740		0.500	
put 760		-0.500	

In the previous section, an example using options on the S&P 500 index illustrated an important consequence of maximizing expected return subject to achieving an minimum level of return under all circumstances: the optimal portfolios result in a probability distribution of pay-offs, with a high probability of obtaining the minimally required pay-off and a small probability of a very high pay-off.

The casino-effect is not a consequence of the way we specified our model. It can be shown that it drives the optimal solution to a larger class of models where expected return maximization is combined with downside risk minimization (the appendix contains a proof of this claim under the assumption of complete markets). From the viewpoint of financial theory, this result may not be surprising. However, since many investors may consider a portfolio with such a pay-off function undesirable, we will suggest a model formulation that enables one to reduce the impact of the casino effect in the optimal solution to the extent that one chooses. This introduces a trade-off to be made by the investor: since the expected pay-off is maximized by casino type pay-off functions, the casino effect can be reduced only at the cost of a lower expected return. In the next section we extend the model with a chance constraint in order to control the casino effect. The results of the alternative model are tested using the same example as in section 2.

3.1 Adding a Shortfall Constraint

Since the casino effect is not a consequence of using LP models, but a consequence of inadequate specification of investment preferences, the question is how investor preferences can be reflected more accurately. Assuming that many investors dislike casino solutions because of the low probability of achieving a satisfactory return, adding shortfall constraints is a natural way to obtain a better specification of investor preferences (see Leibowitz and Henriksson (1989) for a discussion on the use of shortfall constraints in asset allocation models).

We will extend the model by a chance constraint that requires the probability of a return less than a prespecified threshold level to be acceptably small. Here, we will only include a chance constraint on one level of target return. The model, however, can trivially be generalized to include several chance constraints.

Adding constraint (13) to problem (P) in 2.1:

$$\Pr\{V(S; x^p, x^c, y, z) \le \gamma\} \le u, \qquad (13)$$

reflects the requirement of achieving at least a return equal to γ with a minimum probability of u. Assuming lognormally distributed stock index returns, a decreasing state-price density function (Dybvig (1988)) implies that pay-off patterns which are not monotonically increasing in the index, are suboptimal. Using this property, we can reformulate the chance constraint:

$$\begin{aligned}\Pr\{V(S; x^p, x^c, y, z) \le \gamma\} &= \Pr\{S \le V^{-1}(\gamma; x^p, x^c, y, z)\} \\ &= F(V^{-1}(\gamma; x^p, x^c, y, z)).\end{aligned}$$

Now, constraint (13) can be replaced by

$$F(V^{-1}(\gamma; x^p, x^c, y, z)) \le u.$$

Again, applying transformations (by F^{-1} and V, respectively) yields:

$$V(F^{-1}(u); x^p, x^c, y, z) \ge \gamma, \qquad (14)$$

which is a linear constraint in x^p, x^c, y and z. Replacing constraint (11) in problem (P') by constraints (18) and (19), ensures that pay-off functions of feasible portfolios have monotonic pay-offs. In conjunction with (17) and (21), these constraints also ensure that the worst case constraint is always met. Using (14) we now obtain the following LP model:

$$\max_{x^p, x^c, y, z} \; e^{\mu T} y + e^{rT} z + \alpha^{p\prime} x^p + \alpha^{c\prime} x^c \qquad (15)$$

$$\begin{aligned}
p' x^p + c' x^c + y + z &\le 1, && (16)\\
V(0; x^p, x^c, y, z) &\ge \theta, && (17)\\
V(K_1; x^p, x^c, y, z) &\ge V(0; x^p, x^c y, z), && (18)\\
V(K_{i+1}; x^p, x^c, y, z) &\ge V(K_i; x^p, x^c, y, z),\ i = 1, 2, \ldots, n-1, && (19)\\
V(F^{-1}(u); x^p, x^c, y, z) &\ge \gamma, && (20)\\
y + \iota' x^c &\ge 0 \quad , && (21)
\end{aligned}$$

$$x^p, x^c \in \mathbb{R}^n, y, z \in \mathbb{R}.$$

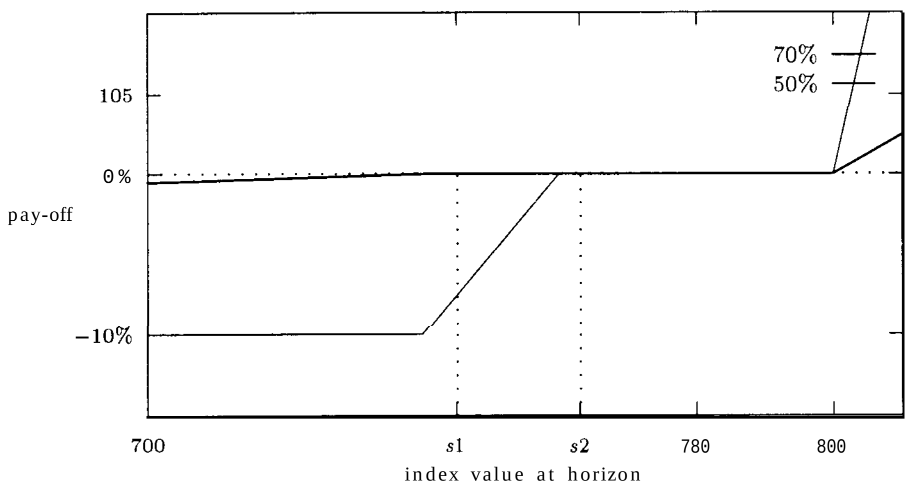

Figure 2: Comparison of pay-off functions

3.2 The S&P 500 Example Continued

In this section we will present computational results that have been obtained from applying the model with a chance constraint (section 3.1) to the S&P 500 data that were presented in section 2.4.

Table 3 presents the solution to the problem that has been discussed in section 2.4, with the additional requirement that the probability of a negative return (a pay-off less than 100) should not exceed 0.5. In conjunction with the monotonicity requirement and the worst case constraint, the chance constraint effectively turns the original worst case pay-off, never less than 90, into a stepwise linear minimum pay-off constraint: at least 90 for index values up to sl (see also figure 2), here, sl is 763, and at least 100 for index values higher than or equal to sl. Although adding a chance constraint should reduce the extent to which optimal pay-off functions reflect the casino effect, one would still expect that solutions are driven by it. Therefore, it is not surprising that the interpretation of the solution to the chance constrained problem is similar to the one without a chance constraint: the 99.63 cash position with 0.5 puts 740 and -0.5 puts 760 constitute the cheapest portfolio that generates the minimally required pay-off. Notice that this portfolio generates a stepwise linear pay-off that is almost identical to the minimum pay-off function. Again, the remainder is invested call 800.

Different levels of the minimum probability of a positive return can lead to portfolios that differ a lot, at least at first sight. Nevertheless the above interpretation to the solution seems to remain valid. As an example, increasing the minimum probability of a positive return to 0.7 gives an optimal portfolio with a holding in the index and holdings in call options, without any put options (see table 1). As can easily be verified, the pay-off equals 90 for an index value equal to 0 and increases with increasing index values to become 100 at an index level of 740,

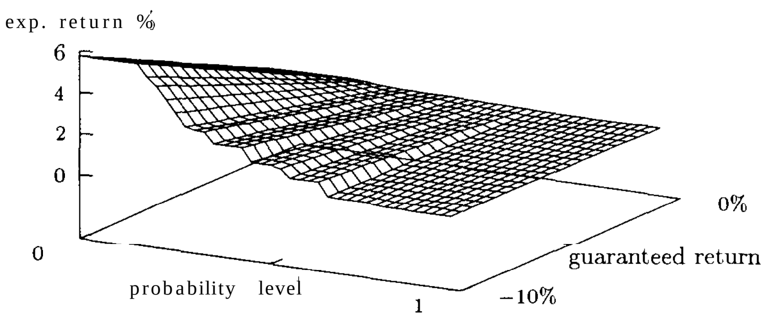

Figure 3: Trading-off expected return, guarantee, 0% chance constraint

which is the index level as of which the pay-off has to be greater than or equal 100 (see $s2$ in figure 2). The cash position in combination with the position in the index and the short position in the call 740 guarantee a pay-off that is sufficient to meet the minimum pay-off requirements at a purchasing price of 99.54. Although it may not be trivial to see, the universe of options and their prices do not allow for composing cheaper portfolios that satisfy the minimum pay-off requirements. The remainder of the budget, 0.46, is invested in the call 800.

Notice that imposing the chance constraints does affect the optimal objective funcition value substantially: the expected return decreases from 5.81% without a chance constraint to 1.77% with the 0.5 minimum probability of a positive return and only 0.63% when this probability is increased to 0.7. Figure 3 pictures the expected return of the optimal portfolio as a function of the level of the worst case return and the minimum probability of a positive return. It provides more insight in the trade-offs that can be made between expected return, the risk of a negative return and the guaranteed downside protection.

4 Summary

In this paper, we have analyzed the trade-off between the expected and guaranteed return of a portfolio, using index options, by means of a linear programming model. The objective of maximizing expected return subject to a minimum return level appeared to imply the optimality of casino policies: the minimum return was achieved with high probability and a very high return only with low probability.

Assuming complete markets, it was shown in the appendix that the casino-effect drives the optimal solution to a large class of models in which expected return maximization is combined with downside risk measures.

Theoretical Black-&holes assumptions imply that optimal pay-off functions increase monotonically in the underlying value. Using this property, we have presented an LP model that serves to maximize the expected return of optioned portfolios, subject to worst case constraints and shortfall constraints. This model enables the investor to maximize expected returns whilst limiting downside risk and controlling the casino effect.

A Appendix

In this section, we will discuss some of the results and claims in the paper in more detail. The first section contains the derivation of necessary conditions for the casino-effect. The second section is devoted to our claim that the casino-effect is not necessarily restricted to the notion of risk that is used in our model, but may be present in many other downside risk optimization models as well.

A.1 Conditions for the Casino Effect

In the absence of transactions costs and/or bid-offer spreads, the put-call parity for non-dividend paying options implies that we may either exclude the put or the call options from the model, since either category can be replicated by a suitable portfolio of the underlying value, the risk-free asset and a position in the other option type, provided there are no arbitrage opportunities. Furthermore, we assume that in any optimal solution, the guaranteed pay-off requirement is satisfied by an appropriate long position in the riskless asset, i.e. an amount of $e^{-rT}\theta$ invested in default-free zero coupon bonds with maturity T. In the absence of arbitrage opportunities, any other portfolio satisfying the guaranteed pay-off requirements would require exactly the same initial investment as in the case of a position solely in the risk-free asset.
Denoting the amount to be invested in the underlying asset by y and the additional investment in the riskless **asset** by z ($z \geq 0$), the LP problem of section 2.3 can be reformulated as follows.

$$
\begin{array}{llll}
(Q) & \max\limits_{x,y,z} \alpha^{c'} x + e^{\mu T} y + e^{rT} z & & \\
& c'x + y + z & = & 1 - e^{-rT}\theta, \\
& -\sum\limits_{i=1}^{j-1}(K_j - K_i)x_i - K_j y - e^{rT} z & < & 0, \qquad j = 1, 2, \ldots, n \\
& -\iota' x - y & \leq & 0, \\
& x \in R^n, y \in R, z \in R+. & &
\end{array}
$$

The casino effect, which was loosely defined in section 2.4, can now be formulated more formally as follows:

Definition A.1 *An optimal solution (x, y, z) to the* **primal** *problem Q exhibits the casino-effect the following conditions hold:*

- $x_i = 0$, i $= 1, 2, \ldots,$ n $- 1$;
- $x_n = (1 - \theta e^{-rT}) c_i^{-1}$;
- $y = 0$;
- $z = 0$.

In other words, the funds that remain after the guaranteed level has been attained, are used to buy the call option with the highest exercise price, presumably the option with the highest expected return. In the following theorem, we present necessary conditions for the casino effect.

Theorem A.2 If *the* casino *solution,* corresponding *to definition A.l, is optimal to problem* Q, *then the following conditions must hold:*

$$\Delta\alpha_{j+1} - \Delta\alpha_j \leq (\Delta c_{j+1} - \Delta c_j)\,\frac{\alpha_n}{c_n}, \qquad i = 2, 3\ , \ldots, \text{ n-1},$$

where $\Delta c_j = c_j - c_{j-1}$, *and* $\alpha_j = \alpha_j - \alpha_{j-1}$.

PROOF: The following proof is constructive and provides directions to derive a set of sufficient conditions. In the proof, we concentrate on the linear programming complementary slackness relations. The necessary conditions above will follow from dual feasibility in a straightforward manner. The LP problem (Q) has the following dual LP problem:

$$\min_{\beta,\gamma,\eta}\,(1 - e^{-rT})\beta \tag{22}$$

$$c_i\beta - \sum_{j=i+1}^{n}(K_j - K_i)\gamma_j - \eta \;=\; \alpha_i, \qquad \text{i} = 1, 2, \ldots, n \tag{23}$$

$$\beta - \sum_{j=1}^{n} K_j\gamma_j - \eta \;=\; e^{\mu T}, \tag{24}$$

$$\beta - e^{rT}\sum_{j=1}^{n}\gamma_j \;\geq\; e^{rT}, \tag{25}$$

$$\beta \in \mathrm{R}, \gamma_j, \eta \in \mathrm{R}^+.$$

Based on this dual and its corresponding primal problem (Q), the following equations are the complementary slackness relations for joint optimality of a candidate primal and dual solution:

$$\gamma_j\left(\sum_{i=1}^{j-1}(K_j - K_i)x_i \quad K_j y - e^{rT}z\right) \;=\; 0, \qquad j = 1, 2, \ldots, n \tag{26}$$

$$\eta\left(\sum_{i1}^{n} x_i + 1\right) \;=\; 0, \tag{27}$$

$$z\left(\beta - e^{rT}(1 + \sum_{j=1}^{n}\gamma_j)\right) \;=\; 0. \tag{28}$$

Substituting the casino solution of definition A.1 into the complementary slackness relations, we observe that any dual optimal solution has **to** satisfy $\eta = 0$ (the term in between brackets in (27) is strictly positive). Omitting the η variable, the dual feasibility constraints (23), (24) and (25) can be reduced to:

$$c_i\beta - \sum_{j=i+1}^{n} (K_j - K_i)\gamma_j = \alpha_i, \quad i = 1, 2, \ldots, n \tag{29}$$

$$\beta - K'\gamma = e^{\mu T}, \tag{30}$$

$$\beta - \gamma e^{rT} \geq e^{rT}, \tag{31}$$

$$\beta \in \mathrm{R}, \gamma_j \in \mathrm{R}^+.$$

The optimal value for β follows directly from (29), as all the γ_j variables vanish for $i = n$:

$$\beta = \frac{\alpha_n}{c_n}. \tag{32}$$

From now on, we will assume that the exercise prices are equidistant, i.e. $K_j - K_{j-1}$, j=2,3,. . . ,n, is constant. Denote this constant distance by ϵ. This assumption is only made to provide to more insight, the proof can easily be adjusted for more general cases, as long as the exercise prices are rational numbers. By consecutive subtracting of equations in (29), we obtain:

$$\Delta c_i\beta + \epsilon \sum \gamma_j = \Delta\alpha_i, \quad i = 2, 3, \ldots, n. \tag{33}$$

This system of equations can be solved by applying backward induction. The optimal values for γ_j satisfy

$$\gamma_n = \epsilon^{-1} (\Delta\alpha_n - \beta\Delta c_n), \tag{34}$$

$$\gamma_j = \epsilon^{-1} \{(\Delta\alpha_j - \Delta\alpha_{j+1}) - \beta(\Delta c_j - \Delta c_{j+1})\}, \quad j = 2, 3, \ldots, n-1. \tag{35}$$

Using the nonnegativity of the dual variables γ_j gives the desired result, after the terms have been rearranged. □

The necessary conditions for the casino effect can be interpreted in the following way. The term $\Delta c_{j+1} - \Delta c_j$ corresponds to the initial costs of a butterfly option strategy, which is a well-known option portfolio which creates a nonnegative pay-off function. Similarly, $\Delta\alpha_{j+1} - \Delta\alpha_j$ refers to the expected gains of this position. Apparently, the profitability of all possible butterfly spreads (measured as the expected gains divided by the costs) should be bounded by the expected return on the call option with the highest exercise price for the casino-effect to hold. This expected return equals the optimal value for β in the dual problem, as is shown by (32).
The conditions in theorem A.2 give us more insight into the forces that drive the casino effect. The intuitively appealing assumption that all the remaining funds are spent to buy the instrument with the highest expected return is just too simple if short-selling is allowed.

A.2 Downside Risk Measures and the Casino Effect

To illustrate why the casino effect may be present in a more general class of downside risk models, we will concentrate on a different problem formulation than the call option model discussed in the previous section. Again, we look at a. one-period economy, where a given initial budget of one dollar should be allocated such that a downside risk measure is minimized, given a predefined expected return level. In contrast with the problems discussed before, we will now restrict ourselves to a finite state space, such that there is a finite number n of possible realizations of the index, which are all equally likely to occur. Markets are assumed to be complete (for instance by assuming that there is a sufficient number of derivative assets traded, see also Ingersoll (1987)), so there is a unique vector p with state prices $p_i, i = 1, 2, \ldots, n$, which prices all possible contingent claims in this economy.

As a consequence of the assumption of market completeness, the asset allocation problem now reduces to allocating money to states of the world, instead of allocating funds to possible assets. In the following analysis, the casino effect will show up in a somewhat different form: it relates to pay-off patterns where all pay-offs above the threshold level that is used in the downside risk measure are concentrated in a single state of the world. To analyze the casino effect, we will use the following minimizing problem.

$$(R) \qquad \min_{\delta_i, v_i} \sum_{i=1}^{n} \delta_i^k \tag{36}$$

$$\sum_{i=1}^{n} p_i v_i \leq 1 \tag{37}$$

$$\sum_{i=1}^{n} v_i \geq \eta n \tag{38}$$

$$v_i + \delta_i \geq \theta, \qquad i = 1, 2, \ldots, n \tag{39}$$

$$v_i, \delta_i \in \text{R+}.$$

In this model, (37) defines the budget equation and (38) restricts the expected return of the pay-off bundle. Equation (39) is necessary to define the deviations of the pay-off in a state below the threshold level θ. Without loss of generality, we will assume that the sequence of state prices p_i, $i = 1, 2, \ldots,$ n, is strictly decreasing. Given suitable choices for the threshold level θ and the expected return target η, there will always be compact feasible region, and, hence, an optimal solution to problem *(R)*. By applying straightforward exchange arguments, it is possible to show that the optimal solution v^* is an increasing sequence. The following theorem states the main result of this section.

Theorem A.3 *Suppose v^* is an optimal solution to problem (R) and that the optimal objective function value is strictly positive. Then at most one of the v_i, $i = 1, 2, \ldots, n$, exceeds the threshold level θ.*

PROOF: Suppose there are at least two states where the pay-offs exceed the threshold level. By monotonicity, it follows

$$\theta < v_{n-1}^* \leq v_n^*.$$

Now construct the following alternative solution w^*:

$$
\begin{aligned}
w_i &= v_i^*, \qquad i = 1, 2, \ldots, n-2, \\
w_i &= v_{n-1}^* - \epsilon, \\
w_i &= v_n^* + \epsilon,
\end{aligned}
$$

with $0 < \epsilon \leq v_{n-1}^* - \theta$. The δ^* remains unchanged. Obviously, w is still feasible to problem (R); the budget constraint equation satisfies

$$\sum_{i=1}^{n} p_i w_i = \sum_{i=1}^{n} p_i w_i + \epsilon(p_n - p_{n-1}) < \sum_{i=1}^{n} p_i v_i^*.$$

Hence, v^* could not have been optimal to (R), which proves the theorem. □

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