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ASSET CLASS ALLOCATION AND DOWNSIDE RISK:

DOES THE INVESTMENT HORIZON MATTER?

by

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ABSTRACT

The main objective of this paper is to analyze within the Mean-Downside Risk (MDR)-framework the relevance of the investment horizon for deriving optimal US asset class allocations. The choice of this risk framework is motivated by its close connection towards the way investors perceive risk and the fact that it is much more general than the often used Mean-Variance (MV) analysis. Unlike the MV-studies of Levy and Gunthorpe (1993) and Thorley (1995) we do not assume returns to follow a random walk. Instead we use a vector autoregressive specification to model the historical time series such that short-term first-order auto- and crosscovariances are preserved. Different from the MV-paper of Lee (1990) is that we explicitly model the short-term first-order auto- and crosscovariances and that we consider more than two asset classes.

Our simulation tests show the weight assigned to stocks to be positively related to the length of the investment horizon. This contradicts the MV-findings of Lee (1990), Levy and Gunthorpe (1993) and Thorley (1995). The relation appears to be most apparent for investors with a low downside-risk aversion. However, even investors whose downside-risk aversion parameter goes to infinity end up with 11% in stocks for long horizons.

This conclusion is based on various assumptions. In order to get insight in the robustness of our results we carried out an extensive sensitivity analysis with respect to the inputs. Only in the situation where we assume a form of investor behavior that is beyond the one that is found in the empirical literature, the aforementioned conclusion is altered. However, the conclusions of the simulation tests are not sensitive to all other changes of the input parameters. Therefore, our results may be considered robust.



1. INTRODUCTION

The idea that the length of the investment horizon is one of the determinants for deriving optimal portfolios is perceived as an indisputable fact among investors. This stems from the widespread practitioner's belief that risk decreases at a higher rate than mean return at increasing horizons. Therefore, risky assets like stocks become almost riskless for very long investment horizons, while prevailing their higher returns. This implies that investors with long investment horizons should allocate more of their wealth to risky assets.

Unlike investment practice, the idea of time diversification is rather controversial and subject to debate among financial theorists. For the one-period case, Thorley (1995) has shown numerically that the relevance of the length of the investment horizon depends on the relative risk aversion of investors and the stochastic properties of risky assets over time. In case investors exhibit a constant relative risk aversion and returns of risky assets are independent and lognormal, the investment horizon is irrelevant for deriving optimal portfolios. However, if returns are not time independent, a property which is implied by a brownian motion, this does not necessarily imply that the investment horizon is relevant. Using historical monthly returns, Thorley shows that the investment horizon is still (more or less) irrelevant for investors.

Lee (1990) and Levy and Gunthorpe (1993) have followed a different approach. In their studies they started from the Mean Variance (MV)-framework. By focusing on the composition of the tangent portfolio for different horizons, they found that at longer investment horizons more wealth should be allocated to bonds. This contradicts the practitioners' view. Since these MV-studies are confined to the tangent portfolio, and because the MV-analysis assumes normal returns and/or quadratic utility - which assumptions are at best approximately valid - the results of the MV-studies are hard to interpret.'

Recently, Bodie (1995) has put the debate concerning the relevance of the investment horizon within an option pricing framework. Essential for his approach is the assumption that the price of a European put option, which ensures an asset against earning less than the risk free rate of return over the investment horizon, is an appropriate measure of that asset's risk. Bodie shows that the cost of shortfall insurance per dollar invested is a monotonically rising function of the length of the investment horizon. Therefore, he concludes that the risk of stocks is also a monotonically rising function of the investment horizon. Bodie's article did elicit a lot of comments (see Cohen (1996), Gould (1996), Merrill & Thorley (1996), Ferguson & Leistikow (1996) and Dempsey, Hudson, Littler & Keasey (1996)). Most of these comments challenge Bodie's conclusions of methodological grounds by arguing that the price of a put option is no valid risk measure for risky assets. A limitation of Bodie's approach, at least for our purpose, is that it only compares two asset classes separately. In this paper however, we will analyse portfolios, taking into account the crosscorrelation term structure, constructed from a universe of five asset classes. Therefore, we need to employ a method for deriving efficient portfolios.

To derive efficient portfolios and to overcome the weaknesses of the MV-framework we will analyze the composition of optimal portfolios for different horizons within a Mean Downside Risk (MDR)-framework. Unlike the MV-framework it is able to capture all possible distributions. Therefore, the MDR-framework is much more general and suitable to analyze the composition of optimal portfolios for both short and long horizons. Moreover, the

Strictly speaking, the normality assumption is not a necessary condition. As long as returns conform to an elliptical distribution it can be shown that the MV-analysis is valid. However, in that case it's better to talk about the Mean-Dispersion framework.

downside risk concept is more closely connected to the risk perception of investors (see, e.g., Markowitz, 1959, 1987; Mao, 1970). It is to be expected that this connection will get closer the coming years, since pension funds and insurance companies are increasingly involved with Asset Liability Management (ALM). Within this framework, current investments are matched with future liabilities. Hence risk is defined as the probability to default on these future liabilities.

Different from Lee(1990) and Thorley (1995), who consider only two asset classes, in our study we will focus on five asset classes: long term government bonds, intermediate term government bonds, long term corporate bonds, small stocks and large stocks, all series for the US and covering the period 1926-1994. Unlike the simulation tests of Levy and Gunthorpe (1993), we do not assume the returns of risky assets to be independent. Instead we explicitly deal with the cross and autocorrelations that are prevalent in the historical return series by fitting a Vector AutoRegressive (VAR) specification. Our tests show, among other things, that for longer investment horizons more wealth should be allocated to stocks. Moreover, since our conclusions are based on various assumptions, we have carried out an extensive sensitivity analysis in order to get insight in the robustness of our results. As will be shown our results are rather robust, since varying the input parameters in several ways does not alter the main conclusions.

This paper is organized as follows. In section 2 we will start with a discussion of the debate on risk and investment horizon. Separately we deal with the arguments of the practitioners and the financial theorists. A short discussion on downside risk is described in section 3. Section 4 describes our data and methodology, followed by a discussion of the results in section 5. In order to investigate the sensitivity of our assumptions we will also present results for alternative input parameters. The paper will be concluded in section 6 with the major conclusions and suggestions for further research.

2. RISK AND THE INVESTMENT HORIZON

Concerning the relevance of the investment horizon for the composition of optimal portfolios there seem to be a controversy between some financial theorists and practitioners. According to the widely held practitioner belief, investors should increase their holdings of risky assets the longer their investment horizon. Support for this proposition is usually given by showing that (i) the probability that risky assets underperform a risk-free alternative is a decreasing function of the investment horizon, and (ii) the standard deviation of the average log-return decreases with a rising investment horizon. Therefore, risky assets become less risky with a rising investment horizon. This argument is known as the ‘diversification of time’ argument.

It can be easily verified that the second argument rests on a fallacy. The average log-return for T years is:

$$\frac{1}{T} \sum_{i=1}^N \ln(1 + r_i) \tag{1}$$

with r_i being the return in period i . If returns are independent and identically distributed then the variance of the average log-returns equals:

$$\sigma_T^2 = \frac{\sigma_1}{T} \quad (2)$$

with σ_T^2 the variance of the T-period average log-return. It follows from (2) that σ_T^2 decreases monotonically if the horizon increases and that for large N it will be considerably lower than σ_1^2 . Upon this analysis it is often argued that the weight of stocks in the asset mix should be a monotonic rising function of the holding period of the investor. However, the essential point that is missed here is that any deviation from the expected average log-return must be summed over a greater number of years. Therefore, the only right way is to look at the standard deviation of the total holding period return or the standard deviation of terminal wealth. This standard deviation usually increases with the length of the holding period.

Whether the first argument - the probability that risky assets underperform a risk-free asset is a decreasing function of the investment horizon - has to be considered a fallacy too is still subject to debate.* According to Kritzman (1994) the argument fails in the limit, meaning that in the limit we will still have the problem that the growing improbability of a shortfall loss is offset by the rising magnitude of a potential shortfall loss. Kritzman supports his proposition with an example without delivering the formal proof. Therefore, the generality of his statement remains unclear.

Contrary to Kritzman however, Thorley (1995) proves that at very long investment horizons risky assets become close to first-order stochastic dominant over the risk free asset. He considers two assets - a risk free fund and a risky fund - and assumes that the risky fund follow a geometric brownian motion with constant mean and constant volatility. From these assumptions it can be derived that the values of the risk free fund (F_s) and the risky fund (G_s) at time s are equal to:

$$F_s = W_0 e^{fs} \quad (3a)$$

$$G_s = W_0 e^{(\mu s + \sigma \sqrt{s} z)} \quad (3b)$$

with:

W_0 = wealth at time 0

f, μ = continuous risk free return and the continuous mean return of the risky fund respectively

σ = volatility of continuous return

z = standard normal variable

From (3a) and (3b) Thorley derives that the shortfall risk of the risky fund over an investment horizon of s periods equals:

$$\Pr ob(G_s < F_s) = 1 - \Phi\left(\frac{\mu - f}{\sigma} \sqrt{s}\right) \quad (4)$$

⁴ Note that unlike the second argument where risk is defined as the standard deviation of returns (total risk measure), here risk is defined as the chance of underperforming the risk free asset (downside or shortfall risk measure). This concept of risk is more closely related to investor's perception of risk. For this reason Thorley (1995) labels this risk measure the practitioner's risk measure.

with Φ being the standard normal cumulative density function. When we assume that the expected return of the risky fund is greater than the expected return of the risk free fund, then it follows from (4) that $\text{Prob}(G_s < F_s)$ approaches zero at very long investment horizons. Upon this analysis Thorley concludes that rational investors are more inclined to choose risky assets the longer their investment horizon.

A more theoretical based treatment of the question whether investment horizon is relevant for deriving optimal portfolios is usually carried out within the Mean-Variance (MV) framework, where the focus is on the composition of the tangent portfolio (Lee, 1990; Levy and Gunthorpe, 1993; Gunthorpe and Levy, 1994) or within utility theory (Thorley, 1995).³ With respect to the first, Lee (1990) focused on the composition of the tangent portfolio for different horizons. The universe of risky assets was confined to two asset classes: stocks and government bonds covering the period 1926-1986. He found that the portfolio weight of stocks in the tangent portfolio increased for horizons up to three years, and declined monotonically for horizons beyond three years. So there is no monotonic relationship between the length of the investment horizon and portfolio weights on the whole domain. According to Lee the mean-reversion pattern in stock returns is primarily responsible for that result. Analyzing the autocorrelation for different horizons revealed that stock returns are mean reverting with a peak at three year periods. In Levy and Gunthorpe (1993) the universe of risky assets has been expanded to common stocks, small stocks, long term corporate bonds, long term government bonds and intermediate-term government bonds, covering the period 1926-1990. Unlike Lee, who derived the tangent portfolio from historical return observation without making specific assumptions regarding the time series properties of returns, Levy and Gunthorpe determined the tangent portfolio for investment horizons of n -years (with $n=2, \dots, 20$). They assumed that returns of risky asset classes follow a random walk. By using annual returns for the different assets over the period 1926-1990, n -year mean returns and n -year variance of the assets can be expressed in terms of 1 -year mean return and the 1-year variance. They found for $n=1$ the proportion of wealth invested in stocks to be equal to 23.4% while it declined monotonically to 2.3% for $n=20$. In Gunthorpe and Levy (1994) similar results were obtained with a universe of risky assets which consisted of five defensive stocks, five neutral stocks and five speculative stocks. The weight of speculative stocks in the tangent portfolio declined in favor of the defensive stocks with an increase in the investment horizon.

The implications of the Levy and Gunthorpe (1993) study seem straightforward: assuming that returns of risky assets conform to a random walk, for longer investment horizons MV-investors should allocate more of their wealth to bonds. The study of Lee (1990) implies a similar conclusion: if returns of risky assets follow a mean-reversion pattern with a peak at three year periods, for horizons beyond three years MV-investors should allocate more of their wealth to bonds while for horizons up to three years the weight of risky assets in the portfolio should increase. However, if we have a closer look at this inference, it reveals an important interpretation problem which is due to the fixation on the tangent portfolio. Every risk averse investors will allocate his wealth over the risk free asset and risky asset in accordance with his risk aversion. By confining the analysis to the tangent portfolio the generalization of the results remain unclear. It may be possible that for a specific utility function the optimal proportion of wealth invested in stocks increases at a rate that more than compensates for the smaller fraction of stocks in the tangent portfolio. This is possible even if the weight of stocks in the tangent portfolio decreases monotonically with an increase in the

Strictly speaking, the MV-analysis is also utility based. However, most studies that employ the MV-approach concentrate on the tangent portfolio instead of some specific utility function.

investment horizon. Only when stocks are swamped from the tangent portfolio, we are certain that the proportion of wealth invested in stocks will be zero for every investor. It must be stressed however, that if stocks are swamped from the tangent portfolio for horizons longer than, say, n years, it does not imply that the relation between the investment horizon and the proportion of wealth invested in stocks is a priori decreasing. The weight of stocks in an investor's optimal portfolio can still be increasing for investment horizons shorter than 4 years.

Thorley (1995) analyses the relation between investment horizon and optimal portfolios within the more general framework of utility theory. He assumes a risk free asset and a risky asset. The stochastic return properties of the risky asset are assumed to be lognormal and in the first instance independent through time. Investors behave according to a derived utility function in which a linear trade-off is made between mean return and variance:

$$U(\mu, \sigma) = \mu - \frac{1}{2} A \sigma^2 \quad (5)$$

with A being the investor's specific risk aversion parameter. He also considers the following class of utility functions:

$$U(W) = \frac{(W - \eta)^{1-\gamma} - 1}{1-\gamma} \quad (6)$$

where η and γ are investor-specific risk-aversion parameters. It can be shown that Relative Risk Aversion coefficient, $RRA(W)$, is equal to:

$$RRA(W) = \frac{\gamma}{1 - (\eta / W)} \quad (7)$$

Hence, if the parameter $\eta=0$, then $RRA(W)=\gamma$ and thus we have the class of iso-elastic utility functions. Loosely speaking, a constant relative risk aversion coefficient means that an investor's attitude toward a percentage loss is the same no matter how much wealth the investor starts with. If $\eta>0$ (<0) then an investor is less (more) risk averse to a percentage loss if he is wealthier.

The optimal weights - given A , and η and γ - are determined for different investment horizons by maximization (5) and the expected value of (6). With respect to (5), Thorley proves analytically that the portfolio weight in risky assets decreases monotonically at longer investment horizons. Although this result confirms the MV-result of Levy and Gunthorpe (1993), it is limited by the fact that they are derived from inconsistent assumptions. The derived utility function in (5) can only be justified by assuming normally distributed returns and an exponential utility function (see, e.g., Sharpe, 1987). Thorley however assumes lognormal distributed returns.

Numerical optimization of the expected value of (6) for $\eta=0$ and $\gamma=8$ reveals that the portfolio weight in risky assets remains constant for every investment horizon.^{4,5} Hence the

⁴ It is not clear from the paper whether Thorley also optimized with other values of γ .

⁵ This result confirms that of Samuelson (1969) and Merton (1969) for the multi-period case (in this paper we consider only the one-period portfolio problem with different lengths of the investment horizon). They prove portfolio weights to be fixed, regardless of the investment horizon, if and only if (i) return distributions are stationary (and thus independent) over time, and (ii) investors have an iso-elastic utility function.

length of the investment horizon is irrelevant. The assumption of independent returns seems rather necessary for the investment horizon irrelevancy. When Thorley used historical returns, the optimal weight in the risky asset is not a constant anymore. It first increases, until a 4-year horizon, and then decreases as the investment horizon increases. For decreasing relative risk aversion ($\eta < 0$), the numerical analysis (with independent returns) shows that the optimal risky asset allocation increases with the investment horizon. It is not surprising that the optimal risky asset weight decreases with the horizon for $\eta > 0$.

Clearly, the relevance of the length of the investment horizon depends on the form of the utility function, the risk aversion parameter and the stochastic properties of returns over time. In the financial economics literature it is often assumed that investors conform to the class of iso-elastic utility functions. For this class of utility functions Friend and Blume (1975) find empirical support. They estimated γ to be less than two. Therefore the relevance of the investment horizon depends on the validity of the random walk assumption.

Concerning this independence assumption, Poterba and Summers (1988) and Fama and French (1988) have shown that US stock returns are mean-reverting over the period 1926-1986, thereby rejecting the random walk assumption. In the studies of Lee (1990) and Thorley (1995) it has been shown that this favors the fraction of wealth allocated to stocks within the tangent portfolio and the optimal risky asset allocation respectively. This increase however is bounded since for holding periods longer than the length of the mean reversal the fraction of stocks in the tangent portfolio respectively the optimal risky asset weight decreases. Hence the investment horizon seems not irrelevant and it looks like that it is connected with mean-reversion patterns.

The foregoing arguments depend on the empirical validity of the iso-elastic utility function. While Friend and Blume (1975) find support for this type of utility function, it has its critics too. Mehra and Prescott (1985) have questioned its validity since in their study to the equity risk premium they found premia that were inconsistent with constant relative risk aversion. If we assume that investors exhibit an increasing or decreasing relative risk aversion - even though these type of utility functions have undesirable properties - the argument for the relevance of the investment horizon becomes even stronger.

3. THE MEAN-DOWNSIDE RISK FRAMEWORK

From the previous section we have to conclude that the relation between risk and investment horizon is still unresolved and surrounded with question marks. Partly, this is due to the limitations of the studies performed so far. The studies of Lee (1990), Levy and Gunthorpe (1993) and part of the study of Thorley (1995) have been performed within the MV-framework. A feature of the MV-analysis is that it assumes normally distributed returns and/or a quadratic utility function. However, for different horizons the normal case cannot be employed. If the one-period returns are normal, the n-period returns cannot, since a product of normal random variables is not normal. Therefore, the application of MV-analysis for different horizons can only be justified by a quadratic utility function. Unfortunately, a quadratic utility function has two well known undesirable properties (Ingersoll, 1987, p. 96): first, all concave quadratic utility functions are decreasing after a certain point, so their validity is bounded, and second, they display increasing absolute risk aversion. As a result, MV-analysis for different horizons can only be approximately valid. Levy and Markowitz (1979), Kroll, Levy and Markowitz (1984) and Markowitz (1959, 1987) have shown that for a wide class of risk averse utility functions the MV-approximation and expected utility are

highly correlated, implying that both methods yield optimal portfolios that are almost the same. However, the MV-approximation deteriorates at longer investment horizons. This notion implies that analyzing the investment horizon within the MV-framework becomes difficult for horizons of two years and beyond.

A more practical limitation of the MV-analysis is its definition of risk: the expected value of the squared deviation between the actual outcome and the expected outcome. Portfolio managers however, are more concerned with not meeting a prespecified target, like for example some kind of broad market index, since their performance is measured relative to an index. And a pension fund will measure its risk relative to its liabilities, that is the probability of not being able to meet the pension obligations to its sponsors.

Because of its practical and theoretical limitations we do not use the MV-analysis in our study. Instead we analyze the relevance of the investment horizon within the MDR-framework. Markowitz (1959, 1987), among others, was the first who set forth the contention that investors frequently associate risk with failure to attain a target return. Mao (1970) reported that risk defined as failing to meet a target level of return is indeed consistent with the practitioner's view of risk. Fishburn introduced in 1977 his so-called a-t model. He showed that for several published empirical studies of risk-taking behavior, below target returns is a fairly well description of risk. The findings of Fishburn were confirmed by Laughum et al (1980).

The a-t model of Fishburn is, like the MV-model, an example of the so-called two attribute risk-return models. In the a-t model return is measured in the same way as in the MV-model, viz. as the expected portfolio return. Instead of the variance as risk measure, however, Fishburn defines the following downside risk (δ) measure (or lower partial moment in the terminology of Bawa, 1978):

$$\delta(r_p; \alpha, t) = E[-\min(0, r_p - t)]^\alpha \quad (8)$$

with

- E = the expectation operator
- r = investment return
- t = prespecified target (return)
- a = measure of risk aversion for returns below the target

The value of a must be non-negative. Values between 0 and 1 implies a risk seeking attitude, $\alpha=1$ implies a risk neutral attitude and $\alpha>1$ implies risk-averse behavior. Loosely speaking, for small values of a an investor is highly concerned for not meeting the target return but has little concern about the size of the deviation, whereas for large values of a there is little concern about small deviations below t but high concern about large deviations below t. In the remainder of the paper the a-t model will be referred to as the MDR-model.

Besides empirical findings on downside risk as appropriate risk measure, the a-t model has also some attractive theoretical properties. First, Fishburn proved that his a-t model is fully consistent with the utility theory of Von Neumann and Morgenstem. The corresponding utility function is given by:

$$U(r) = r - k[-\min(0, r - t)]^\alpha \quad (9)$$

with k the investor's specific downside risk aversion coefficient. Second, the MDR-efficient set belongs to the set of first-order stochastic undominated portfolios. For $\alpha \geq 1$ this holds also with respect to the second order stochastic dominance principle. MV-efficient portfolios however belong to the set of second order undominated portfolios only if return distributions are normal!

4. DATA AND METHODOLOGY

We consider the following US asset classes: Large capitalization Stocks (LS), Small capitalization Stocks (SS), Long-Term Corporate Bonds (LTCB), Long-Term Government Bonds (LTGB) and Intermediate-Term Government Bonds (ITGB). Monthly total returns are available from January, 1926 to December, 1994 (Ibbotson, 1995). With respect to the downside risk measure we take $a=2$ and a target equal to the return on a successive investment in the shortest-term Treasury Bills (TB) having not less than one month to maturity (data also from Ibbotson, 1995). The choice of this target is motivated by the fact that any investor can realize this return without taking any risk. Therefore, investors will perceive any return below that target as a 'perceived loss'. The TB is not included in the set of available asset classes for the following reason. In this paper our primary interest concerns the relation between the composition of optimal MDR-portfolios and long horizons. For long horizons the 1-month TB does not seem a proper asset class. This notion is supported by the empirical observations that for institutions with long horizons the TB does not serve as a strategic asset class.

Since we examine the composition of MDR-optimal portfolios in relation to the investment horizon, we need long-horizon return distributions. One possibility is to use historical holding period returns. Since we examine horizons up to 10 years, using non-overlapping historical returns yields less than 7 observations for the 10-year horizon. Obviously direct use of historical returns is not appropriate. As an alternative the bootstrap methodology could be applied. This method involves drawing (with replacement) series of short-term (monthly in our case) returns from the set of historical returns. These series of returns can be used to determine long-term return distributions. However, the procedure implicitly supposes time-independent returns or a random walk followed by the asset classes. In light of the mean-reversion literature this assumption does not seem very strong. As a consequence the bootstrap procedure is also not suitable. Moreover, studies in the MV-framework showed that if returns of risky assets are assumed to be independent are biased towards bonds. This follows from the fact that long horizon returns are less volatile in mean reverting markets, than in random walk markets. Therefore we do not assume risky assets to follow a random walk. In our simulations we have preserved the short term cross and autocorrelation structure immanent in the historical return series by fitting a VAR specification.

More precisely : if returns are normal then the MV-efficient set plus the minimum variance portfolio is equal to the set of second order undominated portfolios.

Goetzmann and Edwards (1994) defined a VAR representation of the asset class returns in the following manner:

$$r_s = \Omega r_{s-1} + \varepsilon_s \quad (10)$$

with

$$\begin{aligned} r_s &= \text{vector of monthly log returns of the asset classes in period } s \\ \Omega &= \text{matrix of coefficients} \\ \varepsilon_s &= \text{error vector} \end{aligned}$$

They estimated VAR model (10) with historical data and simulated long-horizon returns by drawing (with replacement) from the fitted error vectors. However, VAR model (10) implies a long-run mean of log returns equal to zero. As this is unrealistic, we resort to a slightly different simulation model for asset class returns (see Boender and Romeijn, 1991). Suppose that T successive observations of a series of historical one-month returns on the asset classes are given. Define with respect to the log returns the sample mean vector, the sample covariance matrix and the first-order sample auto- and crosscovariance matrix respectively as (N is the number of asset):

$$\hat{\mu}_i = \frac{1}{T} \sum_{s=1}^T r_{i,s}, \quad i = 1, 2, \dots, N \quad (11a)$$

$$(\hat{\Sigma}_0)_{i,j} = \frac{1}{T} \sum_{s=1}^T (r_{i,s} - \hat{\mu}_i)(r_{j,s} - \hat{\mu}_j), \quad i, j = 1, 2, \dots, N \quad (11b)$$

$$(\hat{\Sigma}_1)_{i,j} = \frac{1}{T} \sum_{s=2}^T (r_{i,s} - \hat{\mu}_i)(r_{j,s-1} - \hat{\mu}_j), \quad i, j = 1, 2, \dots, N \quad (11c)$$

Then given an estimate of Ω , log returns are generated according to:

$$r_s = \hat{\mu} + \Omega(r_{s-1} - \hat{\mu}) + \varepsilon_s \quad (12)$$

by drawing independent ε_s -vectors having mean 0 and covariance matrix equal to $\hat{\Sigma}_0 - \Omega \hat{\Sigma}_0 \Omega'$. Boender and Romeijn (1991) show that if Ω is estimated by $\hat{\Sigma}_1 \hat{\Sigma}_0^{-1}$ then the simulated returns have the following long-run properties:

- (i) the short-term expected log return equals $\hat{\mu}$,
- (ii) the covariance matrix of short-term log returns is $\hat{\Sigma}_0$ and
- (iii) the first-order auto- and crosscovariance matrix of short-term log returns is $\hat{\Sigma}_1$.

In the light of the mean-reversion studies, property (iii) makes the simulation model of Boender and Romeijn particular useful for our situation.

A shortcoming of using sample estimates $\hat{\Sigma}_0$ and $\hat{\Sigma}_1$ for Ω is that all elements of both matrices are used, including the non-significant ones. In order to meet this drawback one can

look at equation (12) as a regression model. Because the model is a system of regression equations, the matrix of coefficients Ω can be estimated with the Seemingly Unrelated Regression (SUR) method (see, e.g., Greene, 1993).⁷ Under the condition that all eigenvalues of Ω are smaller than one in absolute value, a-estimates different from $\hat{\Sigma}_1 \hat{\Sigma}_0^{-1}$ do not change long-run properties (i) and (ii). However, the long run auto- and crosscovariance matrix will be given by $\Omega \hat{\Sigma}_0$ and beforehand this matrix need not be the same as Σ_1 .

We use the backward selection method in order to estimate regression model (12). This means that we start with all explanatory variables and then delete, step by step and starting with the least significant variable, all variables that are not statistically significant at the 5%-level. Table 1 gives the final estimates for the period January, 1926 to December, 1994.⁸ It appears that all determination coefficients, except for Treasury Bills, are very low. This is not surprising, since we know from the large body of literature on the time series behavior of stocks and bonds that there are a lot of factors missing in this model that capture a significant part of the time variation in returns (e.g. Keim and Stambaugh, 1986; Campbell, 1987; Campbell and Shiller, 1988). However, the primary objective of estimating (12) was to preserve the crosscorrelation structure between the asset class returns in the simulation tests and not to derive a model which best gives an ex post explanation of asset returns. Nevertheless, none of the series considered can be treated as pure innovations, since all series are related to its own lagged value and/or the lagged values of other series. Therefore, (12) may be considered an improvement over the random walk model.

TABLE I

Given the estimate of Ω in table 1, we simulated monthly returns by drawing from the ε -vector. Since the distribution is not specified apart from the mean and the covariance matrix, in the first instance we assume it is distributed normally. Sequences of 120 monthly returns are generated and any sequence determines a 1-year, 2-year, . . . , and 10-year return for all five asset classes and the target. For the starting return vector we used the asset class log returns of December 1994. A large number of such sequences, in this paper 1000, approximates the return distributions for each horizon and each asset class. These distributions are used to maximize the expected value of utility function (9) for the following values of k : 5, 10, 25, 50, 100 and infinite. Optimal allocations have been determined using the non-linear optimization routine of Excel (version 5.0). The values of k were chosen such that for the 1-year horizon these numbers yield optimal allocations which are equally dispersed (approximately) on the MDR-efficient frontier. Finally note that by using utility function (9) we avoid the problems inherent with the analysis of tangent portfolios in Lee (1990) and Levy and Gunthorpe (1993).⁹

⁷ Note that no constant should be included in regression model (11).

⁸ The return on treasury bills is included in the regression model because it is used as the target (see table 2).

⁹ In our tests we make use of stochastic targets (the return on successive investments in 1-month treasury bills) instead of a prespecified fixed one. It is not clear whether the correspondence with the utility theory of Von Neumann and Morgenstern, as described in section 3, is still valid in case the target is stochastic.

Table 2 summarizes what we call the base case. In section 5.1 we report and discuss the results of the base case extensively. In order to gain insight in the sensitivity of the results with respect to the inputs, alternative assumptions are examined. These results are reported in section 5.2. We end section 5 by discussing the practical implications of our results (5.3).

TABLE 2

5. EMPIRICAL RESULTS

5.1 Base Case

In the base case we analyze the composition of optimal MDR-portfolios for varying horizons and for six different values of k (the downside risk aversion parameter). The results of this analysis are presented in table 3 and figure 1. A first observation is that the weight of stocks in the optimal MDR-portfolios increases with an increase in the investment horizon. This applies to every k -value. For investors with a low downside risk aversion the weight assigned to stocks is 33% for a one year horizon, while it becomes 100% for horizons of seven years and beyond. For investors with a larger downside risk aversion these weights are lower. For $k=\infty$ we end up with a stock allocation of 11% for a ten year horizon. This contradicts the results of Levy and Gunthorpe (1993), Lee (1990) and Thorley (1995) concerning the relation of holdings in risky assets and the investment horizon within a MV-framework.”

A second observation concerns the rate of change of the portfolio weights in relation to the investment horizon for different k -values. For $k=5, 10$ and 25 the rate of change of the portfolio weights is much more apparent than for $k=50, 100$ and infinite. For $k=\infty$ the portfolio weights seem rather insensitive to the length of the investment horizon. A possible explanation for the fact that the allocation to stocks for $k=\infty$ remains about 10% for all horizons may be our choice of the target. In the base case we assume the target to be equal to successive investments in 1 -month TB. Even with bonds we run the risk of getting short relative to the TB return for any horizon. If we would have set the target equal to zero we would have probably end up with 100% in bonds for $k=\infty$, because for long horizons the bond distribution is completely located above zero, whereas for stocks there remains a small probability of getting below zero. Simulations with a target equal to zero (results not shown here) support this suggestion.

A third observation that follows from table 3 is the fact that some asset classes are dominated. For $k=5$ and 10 the optimal MDR-portfolios are composed of Small Stocks and Intermediate-Term Government Bonds, while for $k=25, 50, 100$ and infinite the optimal MDR-portfolios are expanded with Large Stocks. The assignment to Large Stocks increases with an increase in the length of the investment horizon and a larger value of k . Contrary to Large Stocks the wealth allocation to Small Stocks decreases with a larger value of k , while preserving the property that the allocation to Small Stocks is positively related to the length of the investment horizon. The weight allocated to Long-Term Corporate Bonds and Long-Term Government Bonds remains zero for all k -values and for all horizons.

¹⁰ Recall that their studies are hard to interpret due to **shortcommings** in their tests. While Levy and Gunthorpe (1993) and Lee (1990) confine their analysis to the tangent portfolio, the results of Thorley regarding the MV-analysis are derived from inconsistent assumptions.

In sum, the results of the base case indicate the following main conclusion. Assuming that downside risk is the proper risk measure, every risk averse investor with $a=2$ and a target equal to successive investments in 1-month Treasury Bills should increase his weight in stocks the longer his investment horizon.

TABLE 3 AND FIGURE 1

5.2 Sensitivity analysis

In order to investigate the sensitivity of the base case results, we defined various alternatives to the inputs. Table 4 gives, besides the base case, the alternatives we examined. Each alternative is defined as the base case with one change in one of the inputs (see third column in table 4). Major differences with the results of the base are discussed below.

TABLE 4

Estimating Ω

First we examine the impact of the estimated Ω on the results. The way Ω is estimated only affects the long-run first-order auto- and crosscovariance matrix. In general it is equal to $\Omega \hat{\Sigma}_0$. So, if the matrix Ω is computed by $\hat{\Sigma}_1 \hat{\Sigma}_0^{-1}$ then the first-order auto- and crosscovariance matrix will be $\hat{\Sigma}_1$, i.e., the same as the historical estimated one. When both matrices were compared (results not shown here), the difference appeared to be very small. Hence both simulated return series have the same long run mean, covariance matrix and first-order auto- and crosscovariance matrix. As a result we expect the same results as the base case. Simulating with the alternative Ω confirmed this.

As a second alternative we took the null matrix for Ω . In that case the simulated log return series conform to a random walk. This will reveal the degree of importance of preserving the lagged covariance structure of the asset classes. The simulation results for $\Omega=0$ are presented in figure 2. It appears that - in comparison with the base case (see figure 1) - the differences are small. Except for $k=5$ and $k=10$ the weight assigned to stocks is considerably larger for short horizons. However, the difference between the stock allocation in the base case and the present case is largest for low k -values (i.e., for low downside risk aversion). Further inspection of the optimal MDR-portfolios (results not shown here) reveals that the relative position of Small Stocks compared to Large Stocks is not significantly affected by the random walk assumption. From the simulation results for $\Omega=0$ we may conclude that assuming a random walk in situations where $\Omega \neq 0$, thus the first-order autocorrelations of the assets are nonzero, is valid (the base case), the results are biased towards stocks. This conclusion is in line with the observation of Lo and MacKinlay (1988) who found that short-term stock returns are positively correlated. Positive short-term autocorrelations may enforce the volatility for longer holding periods. Hence, stocks become more downside risky and thus we may expect a lower weight of stocks in the optimal MDR-portfolios.

FIGURE 2

Time varying moments return distribution

In order to check the robustness of our results for time varying moments in the return distributions we have estimated Ω for three non-overlapping periods (see table 4). The years during the Great Depression and World War II (I 926-1946) are characterized by high variability in equity returns. The post-war years 1947-1967 featured high equity returns and low variability in equity as well as debt markets. The third subperiod of the sample, 1968-1994, is dominated by intermediate volatile equity returns.

The results for these three periods did not alter the main conclusion: for every period the assignment to stocks increases with the length of the investment horizon. However, the rate of increase for the period 1947- 1967 differs considerably from the periods 1926-1 946 and 1968-1994. In figure 3A and 3B the results are presented for the periods 1947-1967 and 1968-1994 respectively. Especially for low values for k ($k \leq 25$), the differences are striking. For these values the differences are a monotonic decreasing function of the length of the investment horizon. For $k = \infty$ on the other hand we see that the difference increases with the length of the investment horizon. It follows from the foregoing observations that it is in practice hard to determine *ex ante* optimal MDR-portfolios that are *ex post* optimal as well. The induced estimation risk is a function of the non-stationary in the underlying return distributions.

FIGURES 3A AND 3B

Simulating returns

Asset class returns are simulated with (5) by independently drawing of E-vectors. The E-vector must have mean 0 and covariance matrix $\hat{\Sigma}_0 - \Omega \hat{\Sigma}_0 \Omega'$. For the base case the E-vector is assumed to be distributed normally. However normality is not a necessary assumption and as an alternative we simulated by drawing from the fitted E's (see, e.g., Goetzmann and Edwards, 1994). In appendix C we prove that this sample of E-vectors have also the desired properties: zero mean and a covariance matrix equal to $\hat{\Sigma}_0 - \Omega \hat{\Sigma}_0 \Omega'$. The simulation results yield the following: compared to the base case in figure 1, the weight assigned to stocks is higher for every k -value and every horizon. However, the differences with the base case are most pregnant for low k -values. For example, for $k=5$ the assignment to stocks is 100% for horizons of four years and beyond. Simulating with $\varepsilon \sim N(0, \hat{\Sigma}_0 - \Omega \hat{\Sigma}_0 \Omega')$ and setting k equal to five, results in a 100% assignment to stocks for horizons of seven years and beyond. These results accord with the results for $\Omega=0$ (cf. figure 2).

Downside risk: alpha and the target

The base case value of alpha, 2, has been chosen concordant to the variance which also squares deviations. Alternative values range from slightly larger than 1 to 10. These numbers are based on the findings of Fishburn (1977) who estimated a values ranging from less than 1 to slightly greater than 4. We have analyzed the optimal MDR-portfolios for $\alpha=1.001, 1.5, 3, 4, 5$ and 10.¹¹ The results for values for α equal or smaller than four are similar to those of the base case, and therefore do not alter the main conclusions of the base case.

¹¹ We did not consider $\alpha \leq 1$ for two reasons. First, these values of α imply non-risk averse behavior for returns below the target. The variance as risk measure is only correct in a situation where risk averse behavior is assumed. Since we compare downside risk based decisions to variance based decisions this comparison is only fair if downside risk aversion is assumed. The second reason is more pragmatic. It can be shown that minimizing

A more striking observation is the fact that for a equal or larger than five the weight assigned to stocks decreases with an increase in the investment horizon (see figure 4). This applies to all k -values, except for $k=\infty$. One possible explanation for this inversion may be fact that the target is set equal to the TB return. In that case we may expect the target to be located within the range of possible outcomes for both stocks and bonds. From the nature of stock returns we may also expect the smallest possible outcomes of stocks to be less than the smallest possible outcomes of bonds. For $a=5$ and 10 (hence $\alpha \geq 5$) these deviations of the target contribute substantially to the downside risk measure. Because the deviations are raised to the power of a , large values of a have an adverse effect on downside risk and thus to the allocation of stocks. Even when the probability of realizing an outcome below the target is smaller for stocks, it is possible that this power-effect dominates. This explanation can be verified by setting the target considerably higher than the TB rate. By doing that we may obtain a situation in which for long horizons it becomes almost impossible for bonds to realize the target. In that case we expect that even for low k -values the weight of stocks increases with an increase in the investment horizon. Results with TB+15% did confirm this suggestion.

FIGURE 4

The base case target is not fixed as in the a - t model. Instead we use a stochastic target: the return on a successive investment in the shortest-term TB having not less than one month to maturity. The return on TB can be seen as a sort of minimum return investors can earn. Hence, returns below the TB return contribute to downside risk. In order to examine the influence of the level of the target, annual rates of 2%, 5% and 15% are added to the TB return. Notice that adding a fixed rate of return does not affect the volatility of the target return, only the level of the target.

Increasing the TB with 2, 5 and 15% does not alter the main conclusion of the base case. However, for TB+15% there appears to be a remarkable difference with the base case. For a target equal to the TB return (the base case, see figure 1) it appears that the rate at which the allocation in stocks increased with a rising horizon, decreases for larger values of k . Small additions to the TB do not alter this conclusion. However, as can be seen from figure 5, for TB+1 5% the asset allocation becomes almost insensitive to the downside risk aversion coefficient. For example, when we look at figure 5, we see for $k=25$ that the weight assigned to stocks for an investment horizon of five respectively ten years is equal to 52% and 100%. For $k=\infty$ we find 50% respectively 97%.

In the first instance it seems surprising that it takes a very long time before 100% is invested in stocks (see figure 5). We would expect that the weight assigned to stocks increases at a faster rate than is shown in figure 5. Especially for low k -values and short horizons. This follows from the same reasoning as for $\alpha \geq 5$, i.e., with a high level of the target it is almost impossible for bonds to realize the target. However bonds still play a substantial role in the allocations for most investment horizons. Apparently the power-effect we mentioned earlier is not so pronounced here because of the relative low value of a . Hence bonds are not so downside risky relative to stocks as for the situation with $\alpha \geq 4$.

FIGURE 5

downside risk with $\alpha \leq 1$ for a given mean is a non-convex optimization problem. Preliminary research indicated that results from numerical maximization with $\alpha \leq 1$ are very unreliable.

Finally, the results of the stochastic target are compared to the results of a fixed target of 5% per annum. The simulation results are very similar to the results of the base case. This can be explained as follows. We already noted that TB, TB+2% and TB+5% as a target gives similar results. Since the annualized average monthly return of the TB over the period January, 1926-December, 1994 is 3.6% with a very small volatility (0.9%, annualized), the targets can be assumed fixed over the range 3.6% (=TB return) and 8.6% (=TB+5%). Hence for fixed targets in the 3.686% range, simulation results should not differ. Since the fixed target rate of return of 5% is in the range, it is not surprising that the conclusion regarding this fixed target is similar to the base case.

Available asset classes

Fama and French (1988), Poterba and Summers (1988) and Lee (1990) found long-term mean-reverting behavior (i.e., negative serial correlation) in US equity markets. In particular, Fama and French reported that for 3-5 year horizons around 25% of the price variation is predictable for portfolios of large firms. The percentage is even higher for portfolios of small stocks, viz. 40%. Although not explicitly modeled, we checked for negative serial correlations in our simulated returns. The appendix shows that long-term autocorrelations are not a priori ruled out by simulating model (11). However table 5 makes clear that for Small (and Large) Stocks these numbers are very small. For Intermediate-Term Government Bonds and Treasury Bills we find long-term autocorrelations in the range of 0.15-0.30. Both the Intermediate-Term Government Bonds and Treasury Bills series exhibit characteristics of a random walk with a slowly decaying component.

TABLE 5

In order to investigate the impact on the results of not capturing long-term serial correlations, we considered the asset class allocation problem without Small Stocks, since previous studies indicate that long-term autocorrelations are most present in return series of small stocks. The simulation results without Small Stocks are presented in figure 6. Compared to figure 1 it appears that the differences are small. For $k=5$ the allocation to stocks is larger for investment horizons less than seven years, and for horizons of seven years and beyond the allocations are similar to those of the base case. In case $k=10, 25, 50$ or 100 the weight for stocks is larger for relative short horizons (less than five years in case $k=10$, and nine years for $k=25, 50$ or 100), and smaller for longer horizons. When $k=\infty$ the allocation to stocks is higher for all horizons. One possible explanation for the fact that the results are rather insensitive to the inclusion of Small Stocks is the inability of the simulation test to capture the long-run autocorrelation structure of Small Stocks. This is indicated by the figures in table 5 where we find long-term autocorrelations for Small Stocks that are considerably lower than those reported by Fama and French (1988) and Poterba and Summers (1988). Further research is needed to examine the effects of long-term autocorrelations on the relevance of the investment horizon within the MDR-framework.

FIGURE 6

5.3 Some practical considerations

From the results discussed in 5.1 and 5.2 we may derive the conclusion that for MDR-optimizing investors with long investment horizons and with $\alpha \leq 4$, stocks should be allocated

a substantial weight in the portfolio.¹² Moreover the portfolio weight of stocks should increase with the length of the investment horizon. So far, our main conclusion seems to be in line with the practitioner's point of view and investment practice. Even when we focus our attention on institutional investors our results are robust, since the exclusion of Small Stocks - which asset category is, due to limited liquidity, of limited use - does not alter the main conclusions of the base case.

However, when institutional investors want to turn our simulation results into policy, they will find themselves confronted with some problems we have not dealt with in our analysis. First, even though the investment board of a financial institute may have a long horizon, the portfolio manager's performance is measured (semi-)annually. This agency relation between the investment board and the portfolio manager may introduce a bias in the allocation towards bonds. Second, in case a pension fund or insurance company assigns a large fraction of its holding to stocks, and the stock market is bearish for a couple of years, we may end up in a situation where supervisory institutions will interfere with our asset mix. This would imply that the asset allocation is not path independent. To avoid such a situation pension funds and insurance companies may bias their allocation towards bonds.

Despite these problems we find a close connection of our empirical results with investment practice. Over the last few years professional institutions have shifted their portfolio weights towards stocks, thereby reducing their allocation of real estate and bonds. An often cited argument for this re-allocation is the increased attention for ALM. Within an ALM-framework risk is defined in relation to current and future liabilities. This concept of risk closely matches the downside risk measure. Therefore, the shift from asset management (variance orientation) to asset liability management (downside risk orientation) is probably one of the responsible factors for the re-allocation in favor of stocks.

Another factor that may have encouraged the shift to stocks is the growing competition on the pension market which can be witnessed over the last few years. The liberalization of the pension markets in several countries has boosted competition, which has led to more attention for the seize of the pension premium. In order to guarantee a sufficient number of sponsors, pension funds try to lower the pension premium. One way for accomplishing this goal is to invest more heavily in stocks, so as to enjoy the higher mean returns they offer.

6. CONCLUSIONS

Although investment practice perceives the relevance of the investment horizon as an indisputable fact, we have shown that this conclusion depends very much on the assumption concerning the stochastic properties of the risky assets, and the behavior of investors. The objective of this paper was to analyze the relevance of the investment horizon for deriving optimal portfolios, within the MDR-framework. This choice has been motivated by its close connection to the investors' perception of risk and the fact that it is much more general than the often used MV analysis. Unlike the studies of Levy and Gunthorpe (1993) and Thorley (1995) we did not assume returns to follow a random walk. Instead we used a VAR specification to model the historical time series such, that the short-term auto- and crosscovariances were preserved. Therefore, our results have empirically more meaning.

For downside risk investors with $\alpha=2$ and a target equal to successive investments in treasury bills (the base case, see table 2) we found the weight assigned to stocks to be

¹² Recall that Fishburn (1977) estimated α values ranging from less than 1 to slightly greater than 4. So, the assumption of $\alpha \leq 4$ seems empirically valid.

positively related to the length of the investment horizon. This relation appeared to be most apparent for $k=5, 10$ and 25 . However, even in the situation that $k=\infty$ the allocation to stocks remains 11%. In order to test the robustness of our results, we have carried out an extensive sensitivity analysis with respect to the inputs. Only for $\alpha \geq 5$ we found opposite results: a decreasing weight in stocks with an increase in the investment horizon. Varying the other input parameters by (i) assuming $\Omega=0$, (ii) estimating Ω over three non-overlapping time periods, (iii) drawing with equal probability from the fitted ε 's, (iv) different values of a , (v) increasing the target return, and (v) excluding the Small Stocks from the available asset classes, did not alter the main conclusion of the base case: an increasing weight in stocks for longer horizons.

The simulation results support our contention that the shift towards stocks, within financial institutions, is partly explained by the introduction of the ALM-concept. Within ALM risk is defined in line with the downside risk measure, that is, relative to a prespecified target. Matching our simulation results with contemporaneous investment management would be difficult however. In practice financial institutions operate under a set of restrictions, such as a minimum insurance level, short sales restrictions, etc. In our tests we have not taken these restrictions into account. It is interesting and therefore subject of further research to analyze the influence of these restrictions on the composition of the optimal MDR-portfolios.

Another suggestion for further research is to analyze the characteristics of dynamic strategies within the MDR-framework. This is interesting for at least two reasons. First, it has become clear from the sub-period analyses of optimal MDR-portfolios that simply extrapolating the past into the future may result in ex ante optimal MDR-portfolios that are far from optimal ex post. Therefore, periodic rebalancing may be advocated. Second, few investors will follow long-term buy and hold strategies. Even passive investors will periodically rebalance their portfolio in accordance with their risk attitude or expectations. Therefore, dynamic strategies tend to have more empirical meaning. That is not to say that dynamic strategies are a priori better strategies than the static strategies we have analyzed in this paper. At this moment we study the relevance of the investment horizon for dynamic strategies within the MDR-framework. Results will be presented in a future paper.

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APPENDIX

Mean and covariance matrix of the fitted E's with equal probability

Given an estimate of Ω , the fitted E-vectors from a sample of T periods can be calculated as

$$\varepsilon_s = r_s - \hat{\mu} - \Omega(r_{s-1} - \hat{\mu}), s = 2, 3, \dots, T \quad (A1)$$

Consider the discrete distribution of the fitted &-vectors with equal probability. Then its expected value is

$$\sum_{s=2}^T \frac{1}{T} \varepsilon_s = \frac{1}{T} \sum_{s=2}^T [r_s - \hat{\mu} - \Omega(r_{s-1} - \hat{\mu})] = \hat{\mu} - \hat{\mu} - \Omega\left(\hat{\mu} - \frac{1}{T} r_T - \hat{\mu}\right) \quad (A2)$$

which goes to zero if the sample size goes to infinite.

The covariance matrix of the discrete distribution is

$$\begin{aligned} \sum_{s=2}^T \frac{1}{T} \varepsilon_s \varepsilon_s' &= \frac{1}{T} \sum_{s=2}^T [r_s - \hat{\mu} - \Omega(r_{s-1} - \hat{\mu})][r_s - \hat{\mu} - \Omega(r_{s-1} - \hat{\mu})]' \\ &= \frac{1}{T} \sum_{s=2}^T (r_s - \hat{\mu})(r_s - \hat{\mu})' - \frac{1}{T} \sum_{s=2}^T (r_s - \hat{\mu})(r_{s-1} - \hat{\mu})' \Omega' \\ &\quad - \frac{1}{T} \sum_{s=2}^T \Omega(r_{s-1} - \hat{\mu})(r_s - \hat{\mu})' + \frac{1}{T} \sum_{s=2}^T \Omega(r_{s-1} - \hat{\mu})(r_{s-1} - \hat{\mu})' \Omega' \end{aligned} \quad (A3)$$

which goes for $T \rightarrow \infty$ to

$$\hat{\Sigma}_0 - \hat{\Sigma}_1 \Omega' - \Omega \hat{\Sigma}_1' + \Omega \Sigma_0' \Omega' \quad (A4)$$

Due to $\hat{\Sigma}_1 = \Omega \hat{\Sigma}_0$ expression (A4) reduces to $\hat{\Sigma}_0 - \Omega \hat{\Sigma}_0 \Omega'$. In short, for large T the discrete distribution of fitted E's with equal probability have mean and covariance matrix equal to the desired one's.

Long-run first-order autocorrelation of n-months holding period

We derive an expression for the long-run first-order autocorrelations of the log returns for holding periods longer than the short-term period (1-month in the paper). First we determine the n-month first-order auto/cross covariance matrix at time s (time 0 is the present time), i.e.,

$$\text{Cov}\left(\sum_{i=1}^n r_{s+n+i}, \sum_{j=1}^n r_{s+j}\right) \quad (A5)$$

At any future time z , the log return is (repeated substitution of equation (5))

$$r_z = \hat{\mu} + \Omega^z(r_0 - \hat{\mu}) + \sum_{u=1}^z \Omega^{z-u} \varepsilon_u \quad (A6)$$

Substituting (A6) for $z=s+n+i$ and $z=s+j$ in (A5) and recognizing that r_0 is non-stochastic at time 0 yields

$$\begin{aligned} \text{Cov}\left(\sum_{i=1}^n r_{s+n+i}, \sum_{j=1}^n r_{s+j}\right) &= \text{Cov}\left(\sum_{i=1}^n \sum_{v=1}^{s+n+i} \Omega^{s+n+i-v} \varepsilon_v, \sum_{j=1}^n \sum_{u=1}^{s+j} \Omega^{s+j-u} \varepsilon_u\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \text{Cov}\left(\sum_{v=1}^{s+n+i} \Omega^{s+n+i-v} \varepsilon_v, \sum_{u=1}^{s+j} \Omega^{s+j-u} \varepsilon_u\right) \end{aligned} \quad (A7)$$

Since the ε 's are assumed to be independent through time, expression (A7) reduces to

$$\begin{aligned} \text{Cov}\left(\sum_{i=1}^n r_{s+n+i}, \sum_{j=1}^n r_{s+j}\right) &= \sum_{i=1}^n \sum_{j=1}^n \sum_{u=1}^{s+j} \text{Cov}(\Omega^{s+n+i-u} \varepsilon_u, \Omega^{s+j-u} \varepsilon_u) \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{u=1}^{s+j} \Omega^{s+n+i-u} \text{Var}(\varepsilon_u) (\Omega')^{s+j-u} \end{aligned} \quad (A8)$$

The stationary covariance matrix of ε is defined by

$$\text{Var}(\varepsilon_u) = \Sigma_0 - \Omega \Sigma_0 \Omega', \text{ for all } u \quad (A9)$$

Hence, expression (A8) becomes

$$\begin{aligned} \text{Cov}\left(\sum_{i=1}^n r_{s+n+i}, \sum_{j=1}^n r_{s+j}\right) &= \sum_{i=1}^n \sum_{j=1}^n \sum_{u=1}^{s+j} \Omega^{s+n+i-u} (\Sigma_0 - \Omega \Sigma_0 \Omega') (\Omega')^{s+j-u} \\ &= \sum_{i=1}^n \sum_{j=1}^n \Omega^{n+i-j} \left[\sum_{u=1}^{s+j} \Omega^{s+j-u} (\Sigma_0 - \Omega \Sigma_0 \Omega') (\Omega')^{s+j-u} \right] \\ &= \sum_{i=1}^n \sum_{j=1}^n \Omega^{n+i-j} \left(\Sigma_0 - \Omega^{s+j} \Sigma_0 (\Omega')^{s+j} \right) \end{aligned} \quad (A10)$$

which, if the largest eigenvalue of Ω is smaller than one in absolute value, goes to

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \Omega^{n+i-j} \Sigma_0 \quad (A11)$$

if s goes to infinite (i.e., in the long run).

Along the same line of algebra, it can be shown that the n-month covariance matrix at time s is given by

$$\text{Cov}\left(\sum_{i=1}^n r_{s+i}, \sum_{j=1}^n r_{s+j}\right) = \sum_{i=1}^n \left\{ \sum_{j=1}^i \Omega^{i-j} \left[\Sigma_0 - \Omega^{s+j} \Sigma_0 (\Omega')^{s+j} \right] \right. \\ \left. + \sum_{j=i+1}^n \left[\Sigma_0 - \Omega^{s+i} \Sigma_0 (\Omega')^{s+i} \right] (\Omega')^{j-i} \right\} \quad (\text{A12})$$

Clearly, this goes to

$$\sum_{i=1}^n \left(\sum_{j=1}^i \Omega^{i-j} \Sigma_0 + \sum_{j=i+1}^n \Sigma_0 (\Omega')^{j-i} \right) \quad (\text{A13})$$

in the long run.

The kth element of (A1 1) divided by the kth element of (A13) equals the n-month long-run first-order autocorrelation of asset class k. Taking n = 12, 24, 36,. . . , and 120 gives the long-run 1 -year return, **2-year** return, **3-year** return,. . . , and 1 O-year return autocorrelations (see table 5).

Table 1

SUR estimate of Ω , January 1926 - December 1994

Final estimates of the matrix Ω from model (12) with iterative SUR (828 monthly observations). It has been verified that all eigenvalues of Ω are smaller than one in absolute value, hence the model does not explode. Between parenthesis are t-values. DW is the Durbin-Watson statistic for first-order serial correlation.

	LS(-1)	SS(-1)	LTCB(-1)	LTGB(-1)	ITGB(-1)	TB(-1)	R-square	D	W
LS	0.07 (2.07)				0.15 (2.00)		0.01		1.99
ss	0.21 (4.06)						0.02		1.96
LTCB		-0.01 (-2.26)			0.38 (9.29)		0.06		2.04
LTGB	-0.04 (-3.66)		0.16 (4.42)	-0.23 (-6.75)	0.44 (7.94)		0.05		2.05
ITGB	-0.03 (-4.92)		0.13 (7.64)			0.70 (8.48)	0.08		2.02
TB			-0.00 (-2.37)	0.01 (2.48)	-0.02 (-6.82)	0.99 (126.1)	0.95		2.69

Table 2
Definition of the base case

INPUT	BASE CASE
Method of estimating Ω	Iterative SUR
Historical data used for estimating regression model (12)	Januari, 1926-December, 1994
Simulating time-independent ϵ 's	independent drawings from $N(0, \hat{\Sigma}_0 - \Omega \hat{\Sigma}_0 \Omega')$
Alpha Target	2 total return on successive investements in a 1-month TB
Available asset classes	LS, SS, LTCB, LTGB and ITGB

Table 3
Asset class allocation for various downside risk aversion coefficients
for the base case (see table 2)

k=5					
holding period in years	LS	ss	LTCB	LTGB	ITGB
1		0.33			0.67
2		0.50			0.50
3		0.66			0.34
4		0.79			0.21
5		0.87			0.13
6		0.92			0.08
7		1.00			
8		1.00			
9		1.00			
10		1.00			
k=10					
holding period in years	LS	SS	LTCB	LTGB	ITGB
1		0.18			0.82
2		0.28			0.72
3		0.37			0.63
4		0.44			0.56
5		0.50			0.50
6		0.53			0.47
7		0.60			0.40
8		0.72			0.28
9		0.77			0.23
10		0.84			0.16
k=25					
holding period in years	LS	ss	LTCB	LTGB	ITGB
1	0.03	0.07			0.90
2	0.02	0.13			0.85
3	0.01	0.17			0.82
4	0.03	0.20			0.77
5		0.25			0.75
6		0.27			0.73
7		0.31			0.69
8		0.37			0.63
9		0.41			0.59
10		0.45			0.55

Table 3–Continued

k=50					
holding period in years	LS	s s	LTCB	LTGB	ITGB
1	0.03	0.04			0.92
2	0.04	0.07			0.89
3	0.05	0.08			0.87
4	0.06	0.09			0.84
5	0.03	0.13			0.84
6	0.01	0.15			0.84
7		0.18			0.82
8		0.23			0.77
9		0.26			0.74
10		0.29			0.71
k=100					
holding period in years	LS	SS	LTCB	LTGB	ITGB
1	0.03	0.03			0.93
2	0.04	0.05			0.91
3	0.06	0.04			0.90
4	0.07	0.06			0.87
5	0.06	0.07			0.88
6	0.04	0.08			0.88
7	0.02	0.10			0.88
8	0.02	0.13			0.85
9		0.16			0.84
10		0.18			0.82
k=infinite					
holding period in years	LS	s s	LTCB	LTGB	ITGB
1	0.03	0.02			0.94
2	0.04	0.03			0.93
3	0.06	0.02			0.92
4	0.07	0.03			0.90
5	0.07	0.03			0.91
6	0.05	0.03			0.91
7	0.05	0.04			0.91
8	0.04	0.04			0.91
9	0.05	0.05			0.91
10	0.05	0.06			0.89

Table 4
Definition of the alternatives

INPUT	BASE CASE (table 2)	ALTERNATIVE(S)
Method of estimating Ω	Iteartive SUR	$\hat{\Sigma}_1 \hat{\Sigma}_0^{-1}$ and assuming $\Omega=0$ (i.e., a random walk)
Historical data used for estimating regression model (12)	Januari, 1926-December, 1994	1926-1946, 1947-1967 and 1968-1994'
Simulating time-independent E 's	independent drawings from $N(0, \hat{\Sigma}_0 - \Omega \hat{\Sigma}_0 \Omega')$	independent drawings from fitted E 's of equation (12)'
Alpha Target	2 total return on successive investements in a l-month TB	3, 4, 5, 10, 1.001 and 1.5 TB+2% p.a., TB+5% p.a., TB+15% p.a. and fixed 5% p.a. ²
Available asset classes	LS, SS, LTCB, LTGB and ITGB	base case but no SS available

¹ For each of the last two subperiods the matrix of SUR estimates of Ω is non-positive definite. This implies that the covariance matrix of ϵ is not positive definite. Therefore, for these subperiods we estimated Ω as $\hat{\Sigma}_1 \hat{\Sigma}_0^{-1}$. Based on the sensitivity analysis of the base case with respect to Ω the results should be not significantly **different**.

² For a s-year horizon, $(1+\text{fixed rate of return})^s-1$ is added to the return on the **TB**.

Table 5
Long-run first-order autocorrelation of log returns for various holding periods
for the base case (see table 2)

holding period in years	LS	ss	LTGB	LTGB	ITGB	TB
1	0.02	0.04	0.04	0.05	0.27	0.89
2	0.01	0.02	0.01	0.03	0.29	0.70
3	0.00	0.01	-0.00	0.02	0.27	0.58
4	-0.00	0.01	-0.01	0.02	0.25	0.48
5	-0.00	0.01	-0.01	0.01	0.23	0.41
6	-0.00	0.00	-0.01	0.01	0.20	0.35
7	-0.00	0.00	-0.01	0.01	0.18	0.31
8	-0.00	0.00	-0.01	0.01	0.16	0.27
9	-0.00	0.00	-0.01	0.01	0.15	0.24
10	-0.00	0.00	-0.01	0.01	0.13	0.21

Figure 1
Weight of stocks (=LS+SS) and bonds (=ITGB) of MDR optimal allocations
for the base case (see table 2)

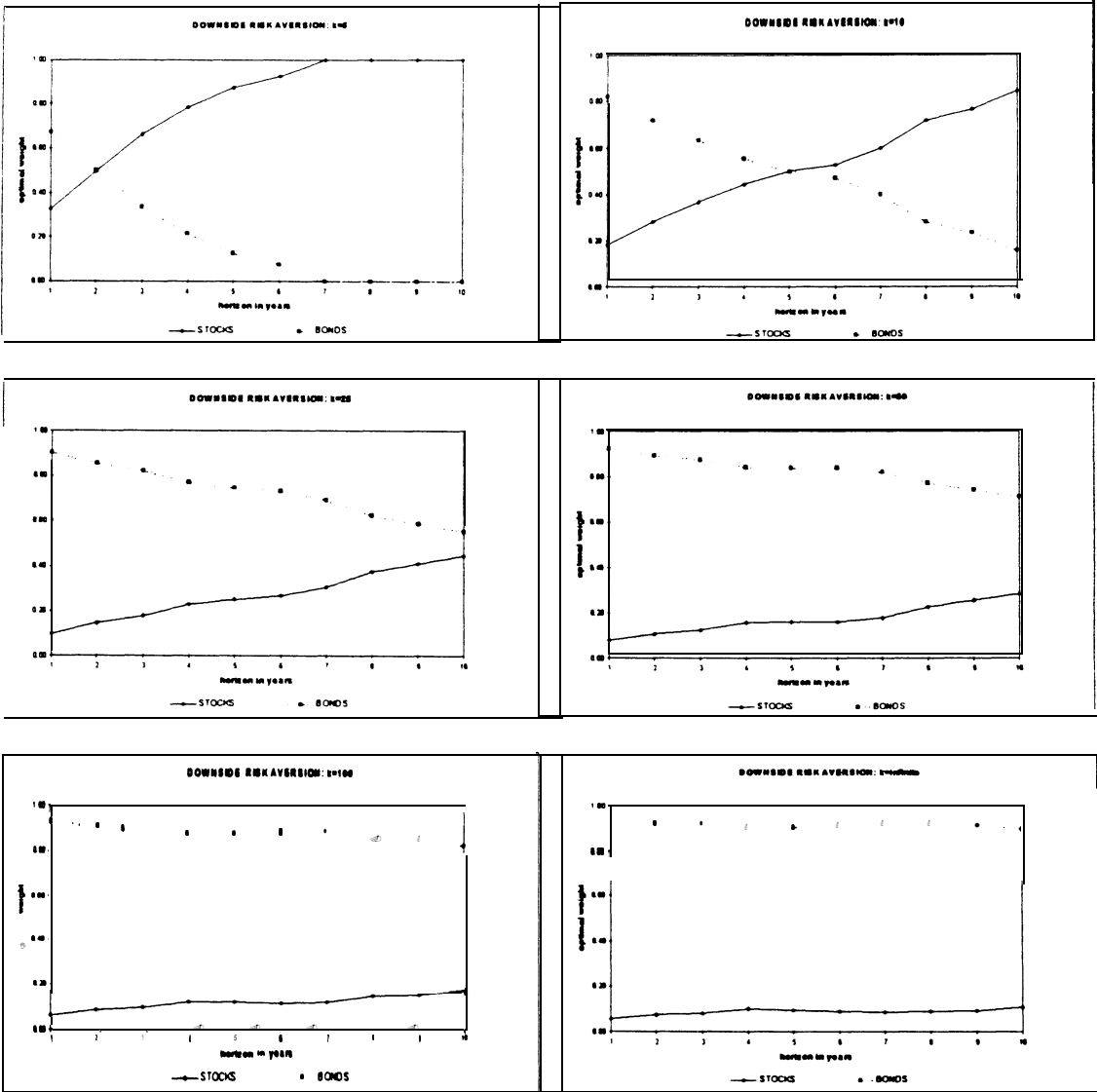


Figure 2
Weight of stocks (=LS+SS) and bonds (=ITGB) of MDR optimal allocations
for $\Omega=0$

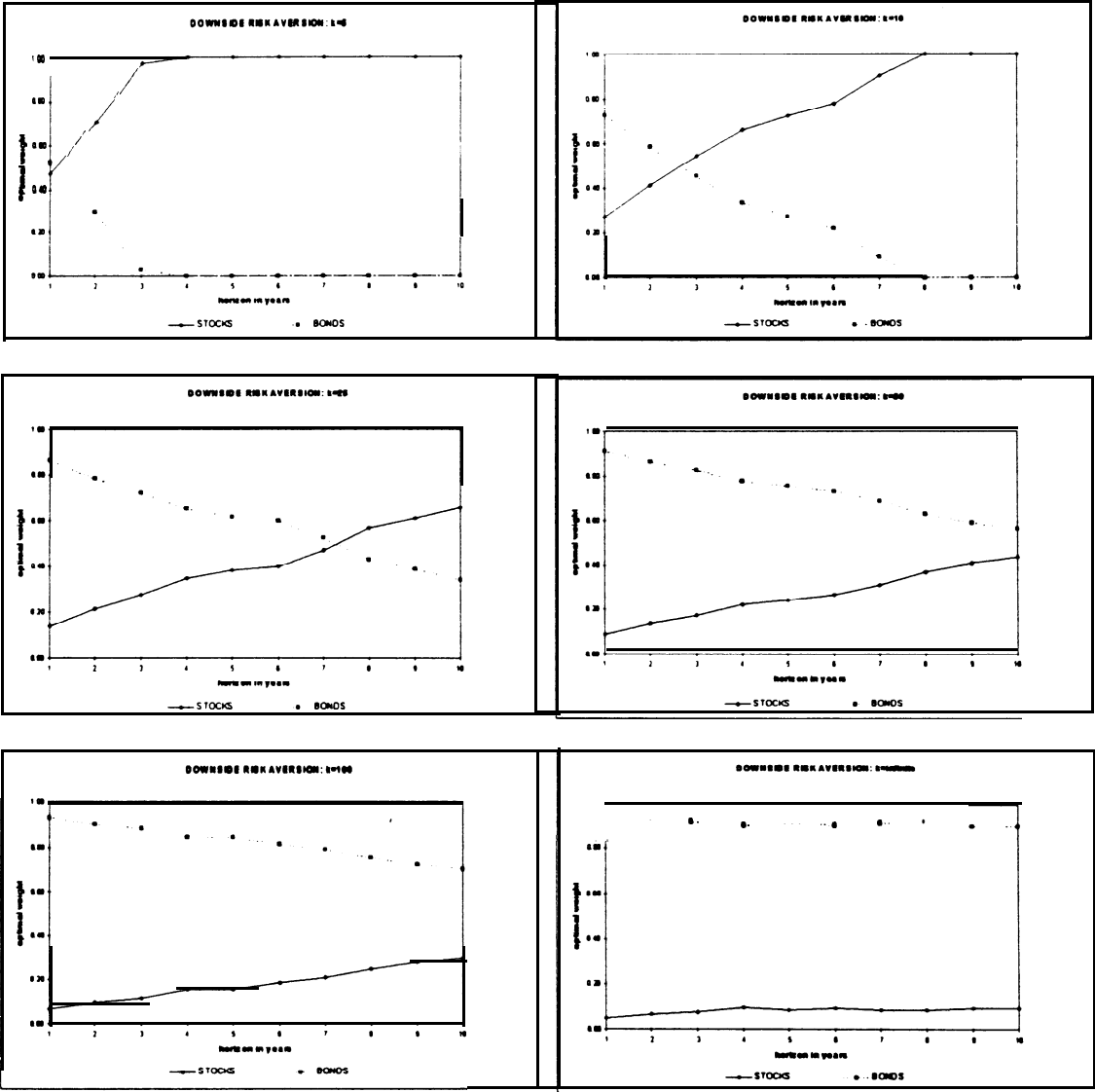


Figure 3A
Weight of stocks (=LS+SS) and bonds (=ITGB) of MDR optimal allocations
for Januari, 1947-December, 1967 sample

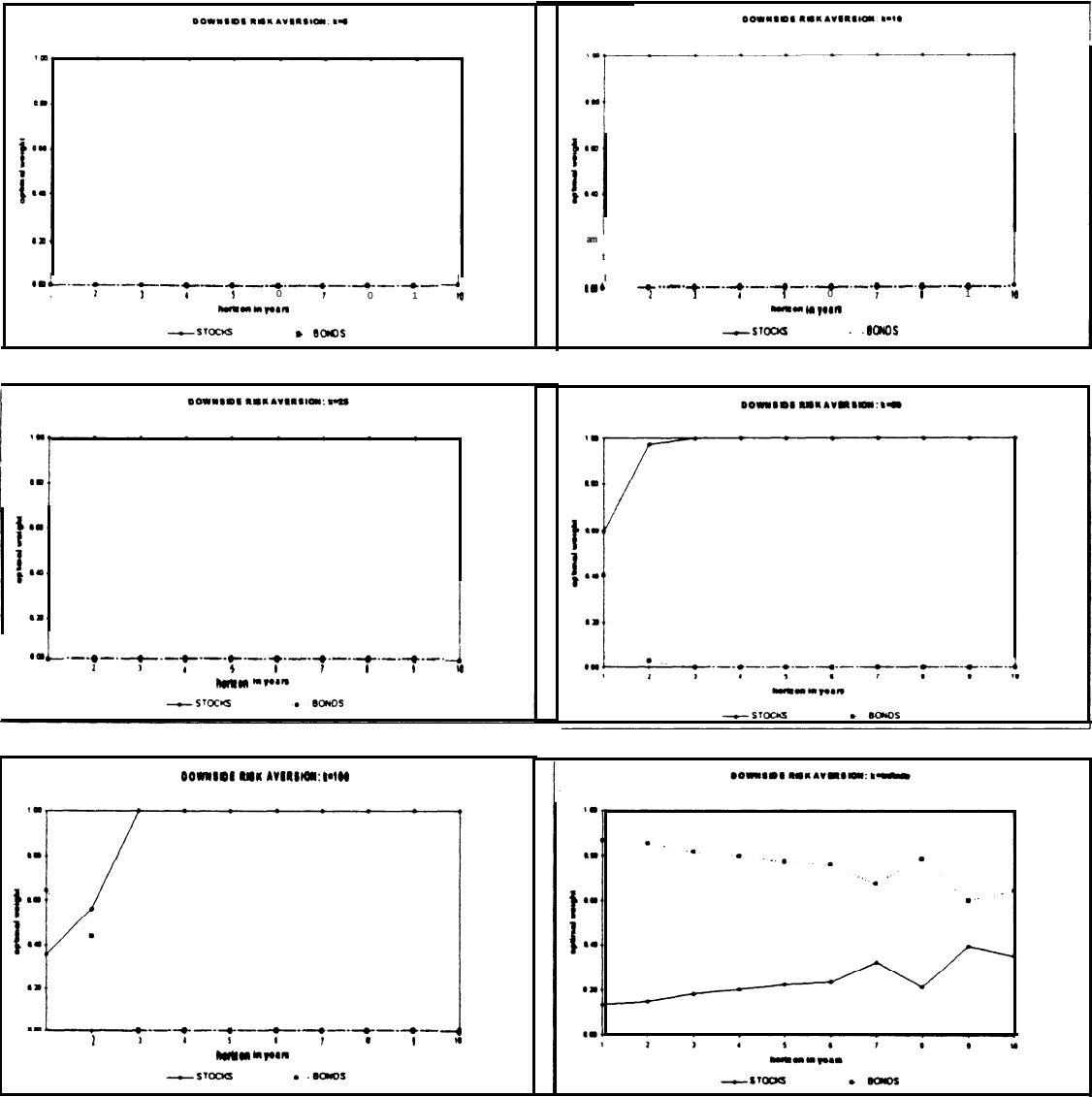


Figure 3B
Weight of stocks (=LS+SS) and bonds (=ITGB) of MDR optimal allocations
for Januari, 1968-December, 1994 sample

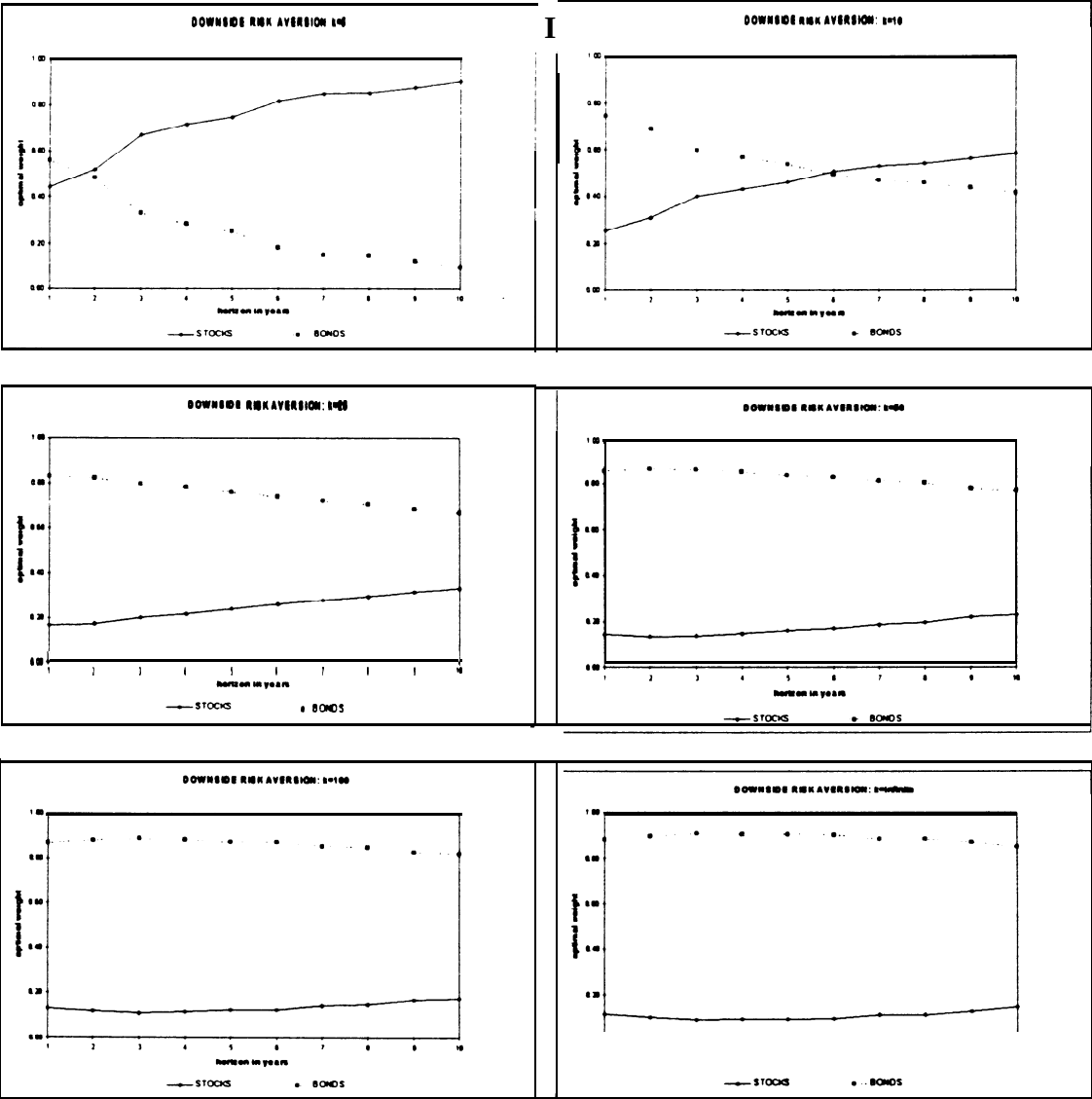


Figure 4
Weight of stocks (=LS+SS) and bonds (=ITGB) of MDR optimal allocations
for $\alpha=5$

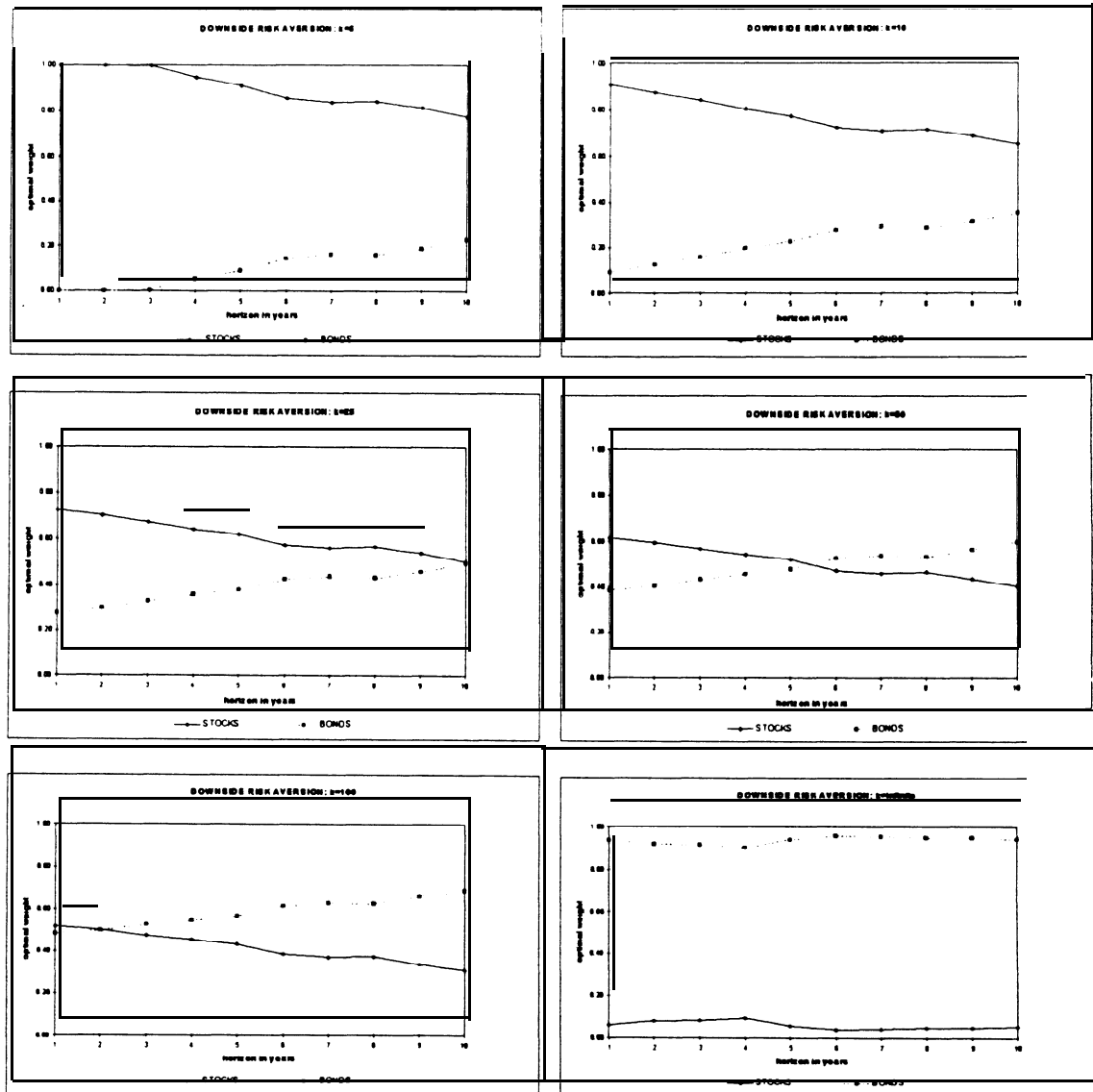


Figure 5
Weight of stocks (=LS+SS) and bonds (=ITGB) of MDR optimal allocations
for target=TB+15%p.a.

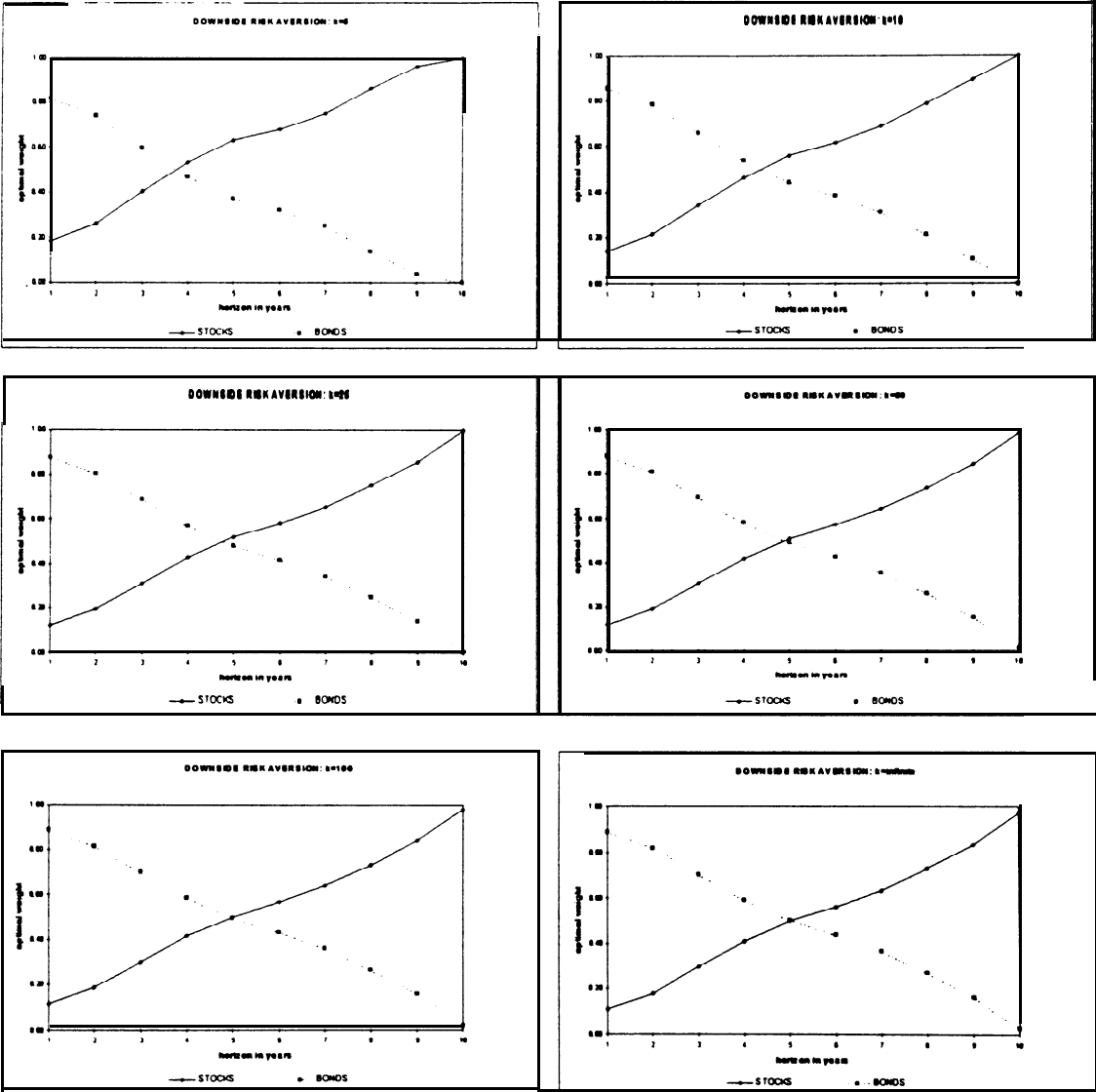


Figure 6
Weight of stocks (=LS) and bonds (=ITGB) of MDR optimal allocations
if no SS available

