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A NOTE ON ERROR BOUNDS FOR APPROXIMATING TRANSITION PROBABILITIES IN CONTINUOUS-TIME MARKOV CHAINS

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A note on error bounds for approximating transition probabilities in continuous-time Markov chains

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Recently Ross [1] proposed an elegant method of approximating transition probabilities and mean occupation times in continuous-time Markov chains as based upon recursively inspecting the process at exponential times. The method turned out to be amazingly efficient for the examples investigated. However, no formal rough error bound was provided. Any error bound even though robust is of practical interest in engineering (e.g., for determining truncation criteria or setting up an experiment). This note primarily aims to show that by a simple and standard comparison relation a rough error bound of the method is secured. Also some alternative approximations are inspected.

1. Introduction. Let $\{X(t), t\geq 0\}$ be a homogeneous continuous-time discrete state Markov chain with transition rate q_{ij} for a transition from state i into state j $(j\neq i)$, so that $q_i = \Sigma_{j\neq i}\, q_{ij}$ is the rate at which it leaves state i. For expository convenience suppose that q_i is uniformly bounded. Denote by P_t the matrix of transition probabilities $P_t(i,j)$ over time t. Then for any $B\geq \sup_i q_i$ we have (cf. Kohlas [2], p...)

$$P_t = \Sigma_{n=0}^{\infty} e^{-tB}[(tB)^n/n!][1+\bar{R}/B]^n = \Sigma_{n=0}^{\infty}[(t\bar{R})^n/n!] \qquad (1)$$

where $\bar{R}$ is the matrix of transition rates $r_{ij}=q_{ij}$ for $j\neq i$ while $r_{ii}= -q_i$. That is, P_t can be thought of as generated by a Poisson process with dominating parameter B which generates jumps and with transition probability matrix $[I + \bar{R}/B]$ upon a jump. (Note that this matrix is stochastic.) This is generally referred to as uniformization. By truncating the Poisson probabilities we hereby have in principle a first straigthtforward method of approximating P_t. A major disadvantage however is its explicit non-linear time dependence. For any different t-value the Poisson probabilities are to be recalculated. We would rather have a successive approximation with a time homogeneous recursion that naturally grows linear in time. To a large extent this has been investigated in Van Dijk [4] for both controlled and uncontrolled Markov processes including jumps and diffusion processes. In essence, it all comes down to a simple comparison result for approximating 'initial value problems' as has long been known in numerical analysis (cf. Lax and Richtmeyer [3]). A simplified version of its application to stochastic matrices will be presented in the next section.

2. <u>Comparison result</u>. Let P_1 and P_2 be stochastic matrices such that for some $\varepsilon>0$:

$$||P_1-P_2|| \leq \varepsilon \tag{2}$$

where $||A||$ stands for the usual supremum norm $||A||=\sup_i \Sigma_j |a_{ij}|$. Then by the telescoping

$$(P_1^n-P_2^n) = \Sigma_{k=0}^{n-1} P_1^k[P_1-P_2]P_2^{(n-1-k)}$$

and the fact that stochastic matrices are supremum-norm preserving (as $||P||=1$ for P stochastic), we immediately conclude

$$||P_1^n-P_2^n|| \leq \varepsilon n \qquad (n\in \;). \tag{3}$$

3. <u>Error bounds</u>. Consider a fixed t, $n\in$ and set $h=t/n$. Let P_h be the transition matrix over time h as in section 1 and let $\overline{P}_h$ be some given stochastic matrix for the purpose of approximation, such that for some $\varepsilon>0$:

$$||P_h-\overline{P}_h|| \leq \varepsilon h. \tag{4}$$

Then from (2) and (3) with $P_1=P_h$ and $P_2=\overline{P}_h$ and the Markov property $P_t=P_{nh}=P_h^n$, we conclude:

$$||P_t-\overline{P}_h^n|| \leq \varepsilon hn = \varepsilon t. \tag{5}$$

In fact, by considering a fixed h and recursively computing $\overline{P}_h^{k+1} = \overline{P}_h(\overline{P}_h^k)$ we thus establish approximations for P_{nh} linearly in time nh. When one is interested only in some expected measure $f(X_t|X_0=i)$ with f a given function, the multiplication and storage of large matrices can be avoided by the recursion

$$\overline{P}_h^{k+1}f(i) = \overline{P}_h(\overline{P}_h^k f)(i)$$

and

$$\overline{P}_h g(i) = \Sigma_j \overline{P}_h(i,j)g(j) \qquad \text{(vi)}$$

so that $\overline{P}_h^n f(i)$ is an approximation of $f(X_t|X_0=i)$ for t=nh. From (5) one immediately concludes the error bound: $\varepsilon t||f||$ with $||f||=\sup_i|f(i)|$. For example, putting f(i)=1 for i∈B and 0 otherwise we so approximate $P_t(i,B)$ as in sections 1 and 2 of Ross [1].

4. Approximations

4.1 Exponential truncation. As a first application, by virtue of the exponential expansion (1) one readily verifies the error bound in (4) with

$$\varepsilon = 2hB^2e^{hB}, \tag{6}$$

where $B=\sup_i q_i$, by using the truncation (Euler approximation)

$$\overline{P}_h = [I + h\overline{R}] \tag{7}$$

with $h\leq B^{-1}$. This linear order in h can be sharpened to an order $O(h^n)$ by

$$\overline{P}_h = \Sigma_{k=0}^n e^{-hB}[(hB)^k/k!][I+\overline{R}/B] = \Sigma_{k=0}^n[(h\overline{R})^k/k!][1-\Sigma_{j=n+1}^{\infty}(hB)^j/j!]. \tag{8}$$

4.2 Ross' approximation. The approximation proposed in Ross [1], as based upon inspection over an exponential time with mean h, yields the approximation matrix:

$$\overline{P}_h = (I-h\overline{R})^{-1} = \Sigma_{k=0}^{\infty}(h\overline{R})^k. \tag{9}$$

Comparing this with the exact exponential expression (1) again, and noting that $(I-h\bar{R})^{-1}$ is a stochastic matrix, we get the error bound

$$2hB^2(1+e^{hB}) \tag{10}$$

which as in (6) is also linear in h. Rather amazingly, however, the approximations of Ross [1] turn out to be much more accurate. Though an error bound of Ross' method is hereby secured, as is the prime intention of this note, further investigation as to a tighter error bound thus remains of interest. As a variation to avoid determining the inverse for large matrices, one could simply truncate the series, say at k=n, which however may lead to a non-stochastic matrix. Nevertheless, by carefully using the telescoping in section 2, one can show that this truncation leads to a deviation from Ross' approximation (9) no more than $e^{tC}O(h^n)$.

Remark. Though this note is restricted to transition probabilities and marginal expectations, extensions in the same spirit can be provided for expected total reward functions up to a given time t (possibly random) such as the mean occupation time up to exiting a given set as in sections 3-5 of Ross [1]. Particularly, in accordance with Van Dijk and Puterman [5] the linear order in time t (possibly as expected time) in (5) can be retained.

References

[1] Ross, S.M. (1987), Approximating transition probabilities and mean occupation times in continuous-time Markov chains. Prob. in the Eng. and Inf. Sciences 1, 251-264.

[2] Kohlas, J. (1982), Stochastic methods of operations research. Cambridge University Press.

[3] Lax, P.D. and Richtmeyer, R.P. (1956), Survey of the stability of linear finite difference equations. Comm. Pure Appl. Math. 9, 267-293.

[4] Van Dijk, N.M. (1984), Controlled Markov processes: time-discretization. CWI Tract 11, Mathematical Center, Amsterdam.

[5] Van Dijk, N.M. and Puterman, M.L. (1985), Perturbation theory for Markov processes with applications to queueing systems. To appear: Adv. Appl. Prob.