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Using Ordinal Information in  
Decision-Making under Uncertainty

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Research Memorandum 1982-12

May 1982



## 1. Introduction

Uncertainty is an essential feature of many decision problems. The uncertainty may relate to various aspects of decision problems :

- uncertainty about the decision-maker's priorities
- uncertainty about the possible states of nature
- uncertainty about the decisions of other actors.

Each of these sources of uncertainty has received attention in analytical approaches to decision making, e.g. multi-objective decision methods, stochastic programming and game theory.

It is important to note that there are various levels of uncertainty. It may range from completely present to completely absent. Thus, where uncertainty about priorities is concerned, we have as one extreme, the complete absence of priority information and as another extreme, complete knowledge of priorities. It is striking that for most fields of decision-making the techniques for the extreme cases seem to be well-developed while the case of intermediate levels of knowledge has received relatively little attention (see Rietveld, 1980, and Nijkamp and Rietveld, 1982).

In this paper, we will pay attention to this intermediate level of knowledge. Examples of this level are : ordinal information, stochastic information and qualitative statements. We will delimit the subject of this paper along two lines : 1) only ordinal information will be taken into account and 2) we will only treat the case of uncertainty about the states of nature. Thus, other types of uncertainty will be assumed to be absent in this paper.

The method presented in this paper is related to a method described by Kmietowicz and Pearman (1981) focussing on extreme expected values. We will show that the method proposed by them may, in certain cases, suggest taking the 'wrong' decision. We will develop a method to deal with ordinal data in such a way that when it is used in combination with the Kmietowicz and Pearman method, this deficiency is removed. In this method mean expected values play a central role.

The extreme expected value method developed by Kmietowicz and Pearman will be presented in Section 2. Section 3 will be devoted to an evaluation of this method. In Section 4, we will develop the mean expected value method, while in Section 5, a number of extensions of the method are discussed.

## 2. The Extreme Expected Value Method

In this section we will present some of the main results of the extreme expected value (EEV) method developed by Kmietowicz and Pearman (1981). The decision problem to be solved by this method has the following features. A decision-maker can select one of a number of alternatives ( $i=1, \dots, I$ ). The pay-off of a certain alternative depends on the state of nature which happens to arise. There can be distinguished  $j=1, \dots, J$  different states of nature. The pay-off of alternative  $i$  given state of nature  $j$  will be denoted as  $X_{ij}$ . It is assumed that ordinary metric information is available about each  $X_{ij}$ . The characteristic assumption made is that ordinal information is available on the probability that each of the future states of nature will arise. Thus, when  $p_j$  denotes the probability that state of nature  $j$  will arise, the available information is of the type :

$$\begin{cases} p_1 \geq p_2 \geq \dots \geq p_J \geq 0 \\ \sum_j p_j = 1 \end{cases} \quad (2.1)$$

Kmietowicz and Pearman use the term 'incomplete knowledge for this kind of information.

The expected value of an alternative  $i$  is equal to :

$$EV_i = \sum_j p_j X_{ij} \quad (2.2)$$

The core of the decision method consists of the determination of the alternative with the maximum or minimum expected value. This task is achieved by determining for each alternative the solution of the following linear programming problem :

$$\begin{cases} \max! \text{ or } \min! & EV_i = \sum_j p_j X_{ij} \\ \text{subject to} & p_1 \geq p_2 \geq \dots \geq p_J \geq 0 \\ & \sum_j p_j = 1 \end{cases} \quad (2.3)$$

It can be shown that due to the specific structure of the constraints in (2.3), the maximum and minimum expected value for alternative  $i$  can be found by examining the following series of  $J$  elements :

$$\left\{ \begin{array}{l} X_{i1} \\ \frac{1}{2} X_{i1} + \frac{1}{2} X_{i2} \\ \frac{1}{3} X_{i1} + \frac{1}{3} X_{i2} + \frac{1}{3} X_{i3} \\ \frac{1}{J} X_{i1} + \frac{1}{J} X_{i2} + \frac{1}{J} X_{i3} + \dots + \frac{1}{J} X_{iJ} \end{array} \right. \quad (2.4)$$

Thus, for each alternative  $i$ , (2.4) yields the maximum and minimum expected value. The most attractive alternative can then be chosen according to the decision criterion adopted: maximin, maximax or a combination of the two.

It is an attractive property of the method that these results can be achieved without the help of any sophisticated programming procedure. Elementary operations suffice to yield the desired result.

Kmietowicz and Pearman show that it is possible to elaborate this method in several directions. For example, when information is available of the type :  $p_1 \geq p_2 + .10$ , this can easily be taken into account. Further, it can be shown that when one wants to determine the alternative with maximum variance, this can be accomplished by examining a series similar to (2.4) so that again there is no need to employ a programming approach.

### 3. Evaluation of the Extreme Expected Value Method

An essential assumption underlying the EEV method is that the decision-maker concerned is able to state his knowledge about the probability of occurrence of the various states of nature in an ordinal way. This is obviously less demanding than the usual way of assigning cardinal values to these probabilities. Kmietowicz and Pearman assume that the ordering of states of nature is complete : for every pair of states of nature the decision-maker is able to indicate which of the two is most probable. This means that one permutation is selected from a set consisting of  $J!$  elements. Ordinal information on the probability of a certain state of nature  $j$  obviously constrains the cardinal values which the corresponding  $p_j$  can assume. This is illustrated in Fig. 1 for the case of three states of nature. Fig. 1.b, which describes the plane  $p_1 + p_2 + p_3 = 1$

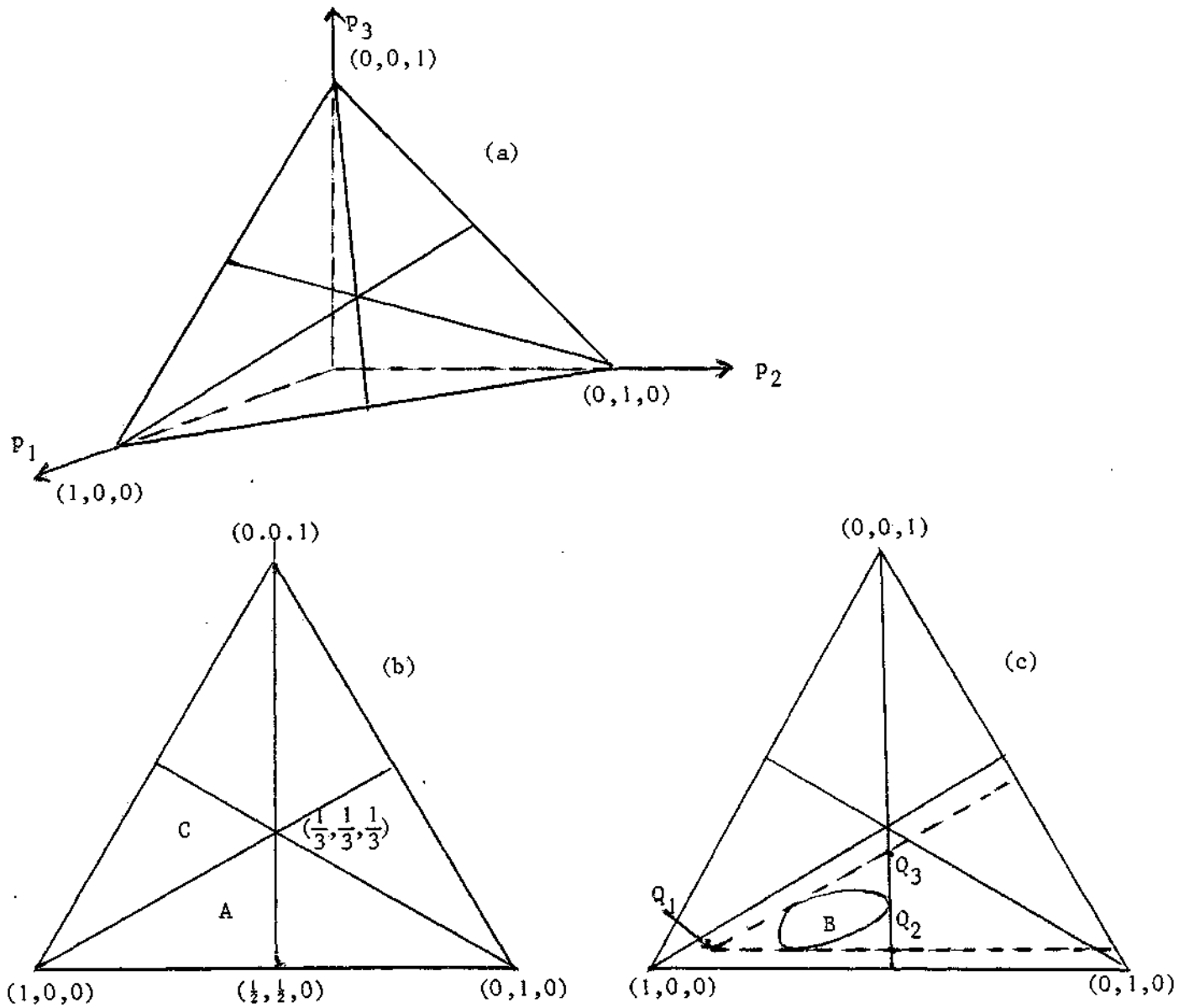


Fig. 1. Probability distribution with three states of nature.

as presented in Fig. 1.a, represents the 6 subsets of  $(p_1, p_2, p_3)$  corresponding with the six permutations. For example, all elements of A correspond with the constraints  $p_1 \geq p_2 \geq p_3$ . Not surprisingly, the corner points of A are the points used in (2.4) to find the maximum and minimum expected value for the alternatives.

It may happen that a decision-maker has more precise information on the probabilities than just  $p_1 \geq p_2 \geq p_3$ . Suppose for example that a decision-maker has the region B in mind as the appropriate set of probabilities (see Fig. 1,c). In that case, the EEV method allows the decision-maker to express his knowledge in the form  $p_1 \geq p_2$ ,  $p_2 \geq p_3 + l_2$  and  $p_3 \geq l_3$ , where  $l_2$  and  $l_3$  are positive constants. In this case, the relevant corner points to be used in (2.4) are  $Q_1$ ,  $Q_2$  and  $Q_3$ .

The EEV method is restrictive in two respects, as far as the provision of information on the probabilities  $p_1, \dots, p_J$  is concerned. First, it does not take into account the possibility that the decision-maker is unable to give a complete ordering of the states of nature. It may happen, for example, that the decision-maker knows that  $p_1 \geq p_2$  and  $p_1 \geq p_3$ , but that it is impossible for him to say whether  $p_2 \geq p_3$  or  $p_3 \geq p_2$ . In this case, two out of six permutations have to be considered:  $p_1 \geq p_2 \geq p_3$  and  $p_1 \geq p_3 \geq p_2$ . This means for Fig. 1.b that both regions A and C remain feasible. Thus the EEV method may demand more information than the decision-maker is able to give.

The second problem with the EEV method is that in some cases it cannot use all information a decision-maker is able to give. For example, information of the type :  $p_2 \leq .4$ , or  $p_1 \geq p_2 + p_4$  is not taken into account in the EEV method.

One may wonder whether the main features of the EEV method can be maintained when the above limitations are to be removed. As far as the first limitation is concerned, if an incomplete ordering of states of nature is allowed, the method can basically be maintained. The only change is that the number of corner points to be inspected increases. The kind of corner points remains the same: they are all of the type :  $(1,0,0)$ ,  $(\frac{1}{2}, \frac{1}{2}, 0)$ ,  $(\frac{1}{2}, 0, \frac{1}{2})$ , etc.

Removal of the second limitation gives rise to larger difficulties. In that case, the maximum and minimum expected values have to be found subject to a constraint set which may have a structure such as (cf. Rizzi, 1982) :

$$\left\{ \begin{array}{l} p_1 \geq p_2 \\ p_2 \geq p_3 + .1 \\ p_3 \geq p_4 \\ p_1 \geq p_2 + p_4 \\ p_2 \leq .4 \\ p_j \geq 0 \quad j = 1, 2, 3, 4 \\ \sum_j p_j = 1 \end{array} \right. \quad (3.1)$$

There does not seem to be a simple way of generating the relevant extreme points of (3.1) which means that one has to use a linear programming procedure. Thus in this case, extreme expected values can no longer be found by elementary operations.

There is another aspect of the EEV method we want to discuss here. Ordinal statements about the probabilities of states of nature 1, 2, ... J such as (2.1) give rise to a set  $B'$  of feasible values for  $p_1, p_2, \dots, p_J$ . This set  $B'$  may be interpreted as an approximation of a set  $B$  the decision-maker has in mind (see e.g. Fig. 1.c). In the EEV method  $B'$  is a convex polyhedron with  $J$  extreme points. It is important to note that the extreme points of  $B'$  are not necessarily elements of  $B$ . Consequently, it is dangerous to base one's selection of an alternative  $i$  exclusively on the values which  $EV_1, \dots, EV_I$  assume in these extreme points.

Obviously, the probability that an interior point of  $B'$  is an element of  $B$  is higher than the probability that an extreme point of  $B'$  is an element of  $B$ . Therefore, it is better that the selection of an alternative is not exclusively based on extreme expected values, but also that information is generated on the values of  $EV_i$  for certain interior points of  $B'$ . In the following section we will show that the centre of gravity of  $B'$  is an attractive candidate for such an interior point.

#### 4. Determination of Mean Expected Values

In Section 3 we found that the relevance of extreme expected values depends on the extent to which  $B'$  is an accurate approximation of  $B$ . Intuitively it is clear that this problem is less acute for the mean expected value<sup>1)</sup> since the latter value depends on the value for  $E_i$  for all elements of  $B'$ , while extreme expected values are determined for just one element of  $B'$ .

Therefore, we will investigate in this section how the mean expected value of an alternative can be determined, given information of the type (2.1). For that purpose we have to know how  $p_1, p_2, \dots, p_J$  are distributed within the constraints (2.1). This information is usually not available and it will be a difficult task for both a decision-maker and an analyst to determine it by some interview technique. Therefore, we will have to assume a certain probability distribution. A usual assumption in this circumstance is that  $p_1, \dots, p_J$  are uniformly distributed within the relevant constraints (see e.g. Zellner, 1971). Thus if  $f_B$  is the real distribution of the probabilities, the decision-maker has in mind, we arrive at  $g_B$  as a uniformly distributed approximation (see e.g. Fig. 2).

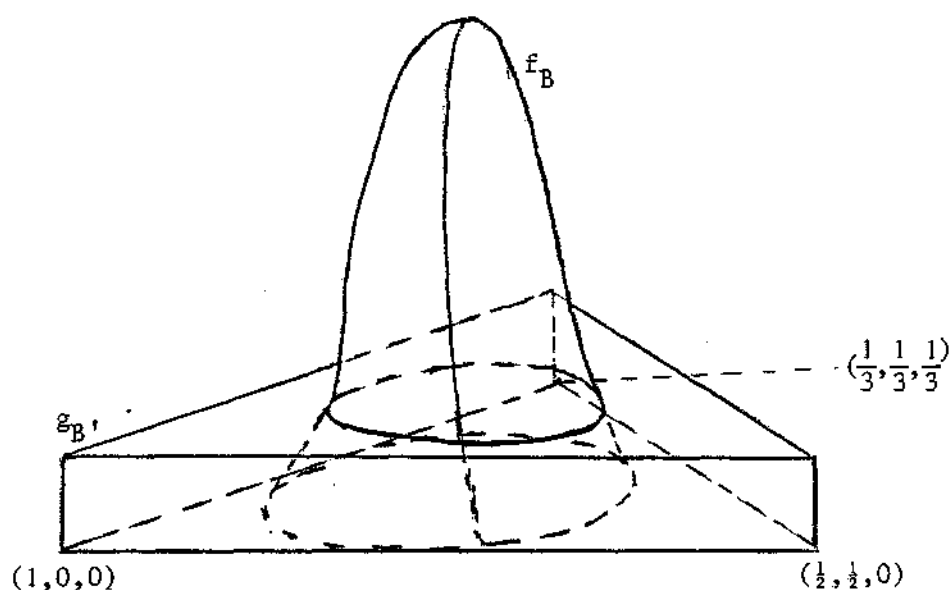


Fig. 2. Probability distributions of states of nature.

As such this may not be a very good approximation, but for the purpose of determining a mean expected value, it may be satisfactory one.

The density function  $g$  (we delete the index  $B$  for the ease of presentation) can be formulated as follows :

$$\left\{ \begin{array}{l} g(p_2, p_3, \dots, p_J) = c \text{ if : } 0 \leq p_J \leq \frac{1}{J} \\ \quad p_J \leq p_{J-1} \leq \frac{1}{J-1} - \frac{1}{J-1} p_J \\ \quad \vdots \\ \quad p_3 \leq p_2 \leq \frac{1}{2} - \frac{1}{2} p_J - \dots - \frac{1}{2} p_3 \\ \quad = 0 \text{ elsewhere} \end{array} \right. \quad (4.1)$$

The left-hand sides of the constraints in (4.1) follow immediately from (2.1). The right-hand sides follow from the condition that  $\sum p_j = 1$ . Once the values of  $p_2, \dots, p_J$  are known, the value of  $p_1$  can be found as  $1 - \sum_{j=2}^J p_j$ .

We will first prove that  $c = (J-1)! J!$

Consider the following expression :

$$A = \int_0^{1/J} \int_{p_J}^{q_{J-1}} \int_{p_{J-1}}^{q_{J-2}} \dots \int_{p_3}^{q_2} (J-1)! J! dp_2 \dots dp_{J-2} dp_{J-1} dp_J \quad (4.2)$$

where

$$q_{J-k} = \frac{1}{J-k} - \frac{1}{J-k} p_J - \frac{1}{J-k} p_{J-1} - \dots - \frac{1}{J-k} p_{J-k+1} \quad (4.3)$$

Proving that  $c = (J-1)! J!$  is equivalent to proving that  $A = 1$ .

After integrating out  $p_2$  we arrive at :

$$A = \int_0^{1/J} \int_{p_J}^{q_{J-1}} \int_{p_{J-1}}^{q_{J-2}} \dots \int_{p_4}^{q_3} \frac{(J-1)! J!}{2} \left( \frac{1}{3} - \frac{1}{3} p_J - \dots \right. \\ \left. \dots - \frac{1}{3} p_4 - p_3 \right) dp_3 \dots dp_J \quad (4.4)$$

Proceeding in the same way we find after integrating out  $p_{J-k-1}$  :

$$A = \int_0^{1/J} \int_{p_J}^{q_{J-1}} \int_{p_{J-1}}^{q_{J-2}} \dots \int_{p_{J-k+1}}^{q_{J-k}} \frac{(J-1)! J!}{(J-k-2)!(J-k-1)!} \cdot (J-k)^{J-k-2} \cdot \left( \frac{1}{J-k} - \frac{1}{J-k} p_J - \dots - \frac{1}{J-k} p_{J-k+1} - p_J \right)^{J-k-2} dp_{J-k} \dots dp_J \quad (4.5)$$

Consequently, the last integration to be carried out is :

$$A = \int_0^{1/J} \frac{(J-1)! J!}{(J-2)!(J-1)!} \cdot J^{J-2} \left( \frac{1}{J} - p_J \right)^{J-2} dp_J \quad (4.6)$$

It is easy to verify that  $A=1$  in (4.6) Q.E.D.

Once the density function  $g$  has been specified, the mean expected value can in principle be determined. It is equal to :

$$E(EV_i) = E\left(\sum_j p_j X_{ij}\right) = \sum_j X_{ij} E(p_j) \quad (4.7)$$

Thus, if we want to know the mean expected value, we have to determine the mean values of  $p_1, \dots, p_J$ .

The mean value of a certain  $p_{J-k}$  is defined as :

$$E(p_{J-k}) = \int_0^{1/J} \int_{p_J}^{q_{J-1}} \dots \int_{p_3}^{q_2} (J-1)! J! p_{J-k} dp_2 \dots dp_{J-1} dp_J \quad (4.8)$$

After integrating out  $p_2, p_3, \dots, p_{J-k-1}$  we arrive at (see also (4.5)) :

$$E(p_{J-k}) = \int_0^{1/J} \int_{p_J}^{q_{J-1}} \dots \int_{p_{J-k+1}}^{q_{J-k}} \frac{(J-1)! J!}{(J-k-2)!(J-k-1)!} (J-k)^{J-k-2} \cdot p_{J-k} \cdot \left( \frac{1}{J-k} - \frac{1}{J-k} p_J - \dots - \frac{1}{J-k} p_{J-k+1} - p_J \right)^{J-k-2} dp_{J-k} \dots dp_J \quad (4.9)$$

For integrating out  $p_{J-k}$  in (4.9) we make use of the fact that the primitive function of  $(a-x)^n x$  is equal to  $\frac{-1}{n+1} (a-x)^{n+1} x - \frac{1}{(n+1)(n+2)} (a-x)^{n+2}$ .

Thus we find that :

$$E(p_{J-k}) = E(p_{J-k+1}) + \int_0^{1/J} \int_{p_J}^{q_{J-1}} \dots \int_{p_{J-k+2}}^{q_{J-k+1}} \frac{(J-1)! J!}{(J-k-1)!(J-k)!} \frac{1}{(J-k)^2} \quad (4.10)$$

$$(J-k+1)^{J-k} \left( \frac{1}{J-k+1} - \frac{1}{J-k+1} p_J - \dots - \frac{1}{J-k-1} p_{J-k+2} - p_{J-k+1} \right)^{J-k}$$

$$dp_{J-k+1} \dots dp_J$$

It is not difficult to show that the second term at the right-hand side of (4.10) is equal to  $\frac{1}{J(J-k)}$ . This implies that  $E(p_J)$  is equal to  $1/J^2$ . Consequently, we have as a final result :

$$\left\{ \begin{array}{l} E(p_J) = \frac{1}{J^2} \\ E(p_{J-1}) = \frac{1}{J^2} + \frac{1}{J(J-1)} \\ \vdots \\ E(p_2) = \frac{1}{J^2} + \frac{1}{J(J-1)} + \dots + \frac{1}{J \cdot 2} \end{array} \right. \quad (4.11)$$

Once  $E(p_2), \dots, E(p_J)$  are determined,  $E(p_1)$  can be found as  $1 - (E(p_2) + \dots + E(p_J))$ . After some elementary operations, it appears that  $E(p_1)$  can be written as :

$$E(p_1) = \frac{1}{J^2} + \frac{1}{J(J-1)} + \dots + \frac{1}{J \cdot 2} + \frac{1}{J \cdot 1} \quad (4.12)$$

so that the structure of (4.11) also holds for  $E(p_1)$ .

In Table 1 we present the outcomes of (4.11) and (4.12) for some selected values of  $J$ .

| J<br>(#states of nature) | mean probabilities |          |          |          |          |          |
|--------------------------|--------------------|----------|----------|----------|----------|----------|
|                          | $E(p_1)$           | $E(p_2)$ | $E(p_3)$ | $E(p_4)$ | $E(p_5)$ | $E(p_6)$ |
| 2                        | .75                | .25      |          |          |          |          |
| 3                        | .61                | .28      | .11      |          |          |          |
| 4                        | .52                | .27      | .15      | .06      |          |          |
| 5                        | .46                | .26      | .16      | .09      | .04      |          |
| 6                        | .41                | .24      | .16      | .10      | .06      | .03      |

Table 1. Mean probabilities of occurrence of various states of nature.

The final result (4.11) has a very simple interpretation: the mean value of  $p_1, \dots, p_J$  is identical to the centre of gravity of the polyhedron defined in (2.1). For example, when  $J=3$ , the extreme points of the polyhedron are  $(1,0,0)$ ,  $(\frac{1}{2}, \frac{1}{2}, 0)$  and  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . The centre of gravity is found by computing the unweighted mean of these points. Thus we arrive at  $(\frac{11}{18}, \frac{5}{18}, \frac{2}{18})$ , which is identical to the outcome of (4.11).

Let us now pay attention to the distribution of the expected value of an alternative given that  $p_1, \dots, p_J$  are distributed according to (4.1). When  $J=3$ , this distribution has a shape such as in Fig. 3. In the EEV method, the attention is exclusively focussed on  $EV_{\min}$  and  $EV_{\max}$  while no attention is paid to the rest of the distribution. Assuming a uniform

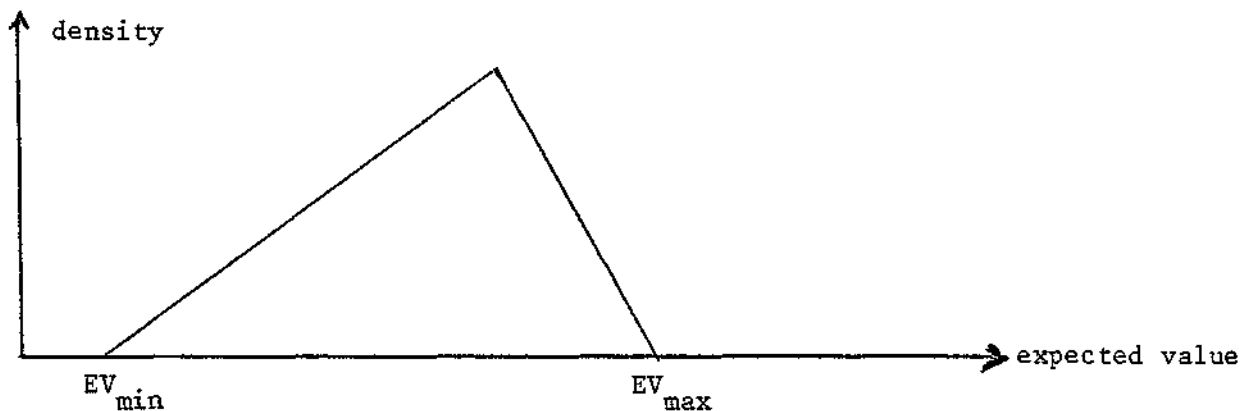


Fig. 3. Probability distribution of the expected value of an alternative.

density function  $g$ , we find that in general the density of EV is high in the interior of the range between  $EV_{\min}$  and  $EV_{\max}$  and low near the extreme values. Thus the probability that EV assumes values near the extremes is in general low compared with the probability that EV assumes values near certain intermediate values. This means that even if  $B'$  is a perfect approximation of  $B$  (so that the extreme points of  $B'$  are indeed elements of  $B$ ) it is questionable to base a decision exclusively on  $EV_{\min}$  or  $EV_{\max}$  since the probability that EV assumes a value near the extremes is low. As we have shown above, it is easy to compute the mean EV (assuming a uniform distribution  $g$ ); therefore we propose using this mean value in addition to  $EV_{\min}$  and  $EV_{\max}$  as a summary indicator of the distribution of  $EV_i$ .

It is not difficult to construct numerical examples in which a decision based on  $EV_{\min}$  and/or  $EV_{\max}$  would be different from a decision based on  $E(EV)$ .

For example, if two alternatives have to be evaluated, given three possible states of nature, where  $p_1 \geq p_2 \geq p_3$  we may have the following pay-offs :

$$\begin{cases} (X_{11}, X_{12}, X_{13}) = (30, 50, -20) \\ (X_{21}, X_{22}, X_{23}) = (23, 21, 85) \end{cases} \quad (4.13)$$

In this case, we find as extreme expected values for the alternatives :

$$\begin{cases} \text{alternative 1 : } EV_{\min} = 20 & EV_{\max} = 40 \\ \text{alternative 2 : } EV_{\min} = 22 & EV_{\max} = 43 \end{cases} \quad (4.14)$$

Hence, if one would use as a selection criterion maximin, maximax or a combination of the two, one would prefer alternative 2 above alternative 1. The mean expected value would give rise to a reverse order, however : 30 for alternative 1 against  $29\frac{1}{3}$  for alternative 2.

## 5. Extensions

The mean expected value approach can be extended into several directions. In this section we will examine the following :

- a. the computation of mean expected values when information of the type  $p_1 \geq p_2 + 1$ , is available
  - b. the computation of the mean variance of pay-offs
  - c. the computation of mean expected values when an incomplete ordering of states of nature is given such as  $p_1 \geq p_2$  and  $p_1 \geq p_3$ .
  - d. the computation of mean expected values when information of the type  $p_1 \geq p_2 + p_4$  is available.
- a. Consider the case that the decision-maker is not only able to give a ranking of states of nature according to their probability of occurrence, but that he can also indicate that the difference in the probability of two subsequent states of nature is at least equal to a certain level.

Thus the available information can be formulated as follows :

$$\left\{ \begin{array}{l} p_1 \geq p_2 + l_1 \\ p_2 \geq p_3 + l_2 \\ \vdots \\ p_{J-1} \geq p_J + l_{J-1} \\ p_J \geq l_J \\ \sum_j p_j = 1 \end{array} \right. \quad (5.1)$$

where the levels  $l_1, \dots, l_J$  are non-negative. Obviously, some of them may be equal to zero. When one assumes that the probabilities are uniformly distributed, one arrives at the following density function :

$$\left\{ \begin{array}{l} g(p_2, \dots, p_J) = c \\ \text{if :} \quad \begin{array}{l} l_J \leq p_J \leq \frac{1}{J} \{1 - l_1 - 2l_2 - 3l_3 - \dots - (J-1)l_{J-1}\} \\ p_J + l_{J-1} \leq p_{J-1} \leq \frac{1}{J-1} \{1 - p_J - l_1 - 2l_2 - \dots - (J-2)l_{J-2}\} \\ p_{J-1} + l_{J-2} \leq p_{J-2} \leq \frac{1}{J-2} \{1 - p_J - p_{J-1} - l_1 - 2l_2 - \dots - (J-3)l_{J-3}\} \\ \vdots \\ p_3 + l_2 \leq p_2 \leq \frac{1}{2} \{1 - p_J - p_{J-1} - \dots - p_3 - l_1\} \end{array} \\ = 0 \quad \text{elsewhere} \end{array} \right. \quad (5.2)$$

It can be proved along the same lines as in Section 4 that :

$$c = \frac{(J-1)! J!}{(1-l_1 - 2l_2 - \dots - Jl_J)^{J-1}} \quad (5.3)$$

if  $\sum_{j=1}^J j l_j < 1$ . If  $\sum_{j=1}^J j l_j = 1$ , the values of  $p_1, \dots, p_J$  can be exactly determined. If  $\sum_{j=1}^J j l_j > 1$ , no values of  $p_1, \dots, p_J$  can be found which satisfy the constraints (5.1).

For the computation of  $E(p_1, \dots, p_J)$ , we can make use of the same procedure as adopted in Section 4. The final result is :

$$\begin{aligned}
E(p_J) &= 1_J + (1 - \sum_{j=1}^J j 1_j) \frac{1}{J^2} \\
E(p_{J-1}) &= 1_J + 1_{J-1} + (1 - \sum_{j=1}^J j 1_j) (\frac{1}{J^2} + \frac{1}{J(J-1)}) \\
&\vdots \\
E(p_1) &= (1_J + 1_{J-1} + \dots + 1_1) + (1 - \sum_{j=1}^J j 1_j) (\frac{1}{J^2} + \frac{1}{J(J-1)} + \dots + \frac{1}{J \cdot 1})
\end{aligned} \tag{5.4}$$

Thus one arrives at a structure similar to (4.11). Here again it can be proved that  $E(p_1, \dots, p_J)$  is the centre of gravity of the corresponding polyhedron (5.1).

- b. Thus far we have focussed on the expected value of the pay-off of an alternative. In addition to this expected value, decision-makers may also be interested in the variance of the pay-offs of an alternative. Given the pay-offs  $X_{i1}, \dots, X_{iJ}$  of alternative  $i$  one might directly compute the variance as :

$$W_i = \sum_{j=1}^J \frac{1}{J} (X_{ij} - \bar{X}_i)^2 \tag{5.5}$$

where  $\bar{X}_i$  is the mean pay-off of alternative  $i$ . This approach is not entirely satisfactory, however, since in (5.5) no attention is paid to the information that some states of nature are more probable than others. Therefore, (5.5) has to be replaced by :

$$\begin{cases} V_i = \sum_{j=1}^J p_j (X_{ij} - \bar{X}_i)^2 \\ \bar{X}_i = \sum_{j=1}^J p_j X_{ij} \end{cases} \tag{5.6}$$

From (5.6) we may derive :

$$V_i = \sum_{j=1}^J p_j X_{ij}^2 - (\sum_{j=1}^J p_j X_{ij})^2 \tag{5.7}$$

Kmietowicz and Pearman show that when one is interested in the extreme values of  $V_i$  given the constraints (2.1), an approach similar to (2.4) can be used. We will show here that along the same lines as in Section 4 expressions can also be found for the mean variance once uniformly distributed  $p_1, \dots, p_J$  are assumed.

Obviously, for this purpose, we have to find expressions for  $E(p_j^2)$  and  $E(p_j \cdot p_m)$ . The derivation of these expressions is a rather tedious exercise. It is in most respects similar to the derivation of  $E(p_j)$ . However, a notable difference is the higher complexity of the primitive function for (4.9). When one wants to determine  $E(p_j^2)$ , one has to find the primitive function of  $(a-x)^n x^2$ .

This function is :

$$= \frac{1}{n+1} (a-x)^{n+1} x^2 - \frac{2}{(n+1)(n+2)} (a-x)^{n+2} x - \frac{2}{(n+1)(n+2)(n+3)} (a-x)^{n+3} \quad (5.8)$$

The final results are :

$$\left\{ \begin{array}{l} E(p_j^2) = \frac{2}{J^3(J+1)} \\ E(p_{J-1}^2) = E(p_J^2) + \frac{2}{(J-1)J^2(J+1)} + \frac{2}{(J-1)^2 J(J+1)} \\ E(p_{J-2}^2) = E(p_{J-1}^2) + \frac{2}{(J-2)J^2(J+1)} + \frac{2}{(J-2)(J-1)J(J+1)} + \frac{2}{(J-2)^2 J(J+1)} \\ \vdots \\ E(p_1^2) = E(p_2^2) + \frac{2}{1 \cdot J^2(J+1)} + \frac{2}{1 \cdot (J-1)J(J+1)} + \dots + \frac{2}{1^2 J(J+1)} \end{array} \right. \quad (5.9)$$

and :

$$\left\{ \begin{array}{l} E(p_{J-k} \cdot p_{J-m}) = E(p_{J-k+1} \cdot p_{J-m}) + \frac{1}{(J-k)(J-m)J(J+1)} + \frac{1}{(J-k)(J-m+1)J(J+1)} \\ \quad + \dots + \frac{1}{(J-k)(J-1)J(J+1)} + \frac{1}{(J-k)J^2(J+1)} \\ \text{for } k < m. \end{array} \right. \quad (5.10)$$

Let  $z_{jj}$ , denote  $E(p_j p_j)$ , and let  $Z$  be the  $J \times J$  matrix with elements  $z_{jj}$ , then we find for  $J = 2, 3$  and  $4$ , respectively :

$$J = 2 \quad Z = \frac{1}{12} \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix} \quad (5.11)$$

$$J = 3 \quad Z = \frac{1}{216} \begin{bmatrix} 85 & 34 & 13 \\ 34 & 19 & 7 \\ 13 & 7 & 4 \end{bmatrix} \quad (5.12)$$

$$J = 4 \quad Z = \frac{1}{2880} \begin{bmatrix} 830 & 386 & 200 & 84 \\ 386 & 230 & 116 & 48 \\ 200 & 116 & 74 & 30 \\ 84 & 48 & 30 & 18 \end{bmatrix} \quad (5.13)$$

Several properties can be shown for the  $Z$  matrix, for example,  
 $\sum_{j,j'} z_{jj'} = 1$  and  $\text{trace}(Z) = \frac{2}{J+1}$ .

When one computes the extreme and mean values of the variance of the pay-offs given in the numerical example (4.13) one obtains :

$$\left\{ \begin{array}{l} \text{alternative 1 :} \\ \quad \text{min. variance} = 0 \quad \text{mean variance} = 677,7 \quad \text{max. variance} = 866,7 \\ \text{alternative 2 :} \\ \quad \text{min. variance} = 0 \quad \text{mean variance} = 659,3 \quad \text{max. variance} = 882,7 \end{array} \right. \quad (5.14)$$

Thus, although the maximum variance in alternative 2 is higher than in alternative 1, the mean variance is higher in alternative 1 than in alternative 2.

- c. When the decision-maker is not able to give a complete ordering of states of nature with respect to their probability of occurrence, one may proceed as follows. Generate all permutations which are in agreement with the available information. For example, if we know that

$$\left\{ \begin{array}{l} p_1 \geq p_2 \geq p_4, \text{ we arrive at :} \\ p_1 \geq p_2 \geq p_4 \geq p_3 \\ p_1 \geq p_2 \geq p_3 \geq p_4 \\ p_1 \geq p_3 \geq p_2 \geq p_4 \\ p_3 > p_1 > p_2 > p_4 \\ \sum p_j = 1, p_1, \dots, p_4 \geq 0 \end{array} \right. \quad (5.15)$$

In general, we obtain in this way  $N$  possible subsets  $S_n$  ( $n=1, \dots, N$ ) of the set of probabilities. Note that  $1 \leq N \leq J!$ . When we assume again a uniform distribution of  $p_1, \dots, p_J$  among the subsets  $S_m$ , the appropriate density function is :

$$\left\{ \begin{array}{ll} g_N(p_1, \dots, p_J) = \frac{(J-1)!J!}{N} & \text{if } (p_1, \dots, p_J) \in S_1 \cup S_2 \dots \cup S_N \\ = 0 & \text{elsewhere} \end{array} \right. \quad (5.16)$$

The expectation of an arbitrary  $p_j$  can be found as follows :

$$\begin{aligned} E(p_j \mid (p_1, \dots, p_J) \in S_1 \cup S_2 \cup \dots \cup S_N) = \\ = \sum_{n=1}^N E(p_j \mid (p_1, \dots, p_J) \in S_n) \end{aligned} \quad (5.17)$$

since  $S_1, S_2, \dots, S_N$  are non-overlapping. Thus  $E(p_1, \dots, p_J)$  given  $g_N$  can simply be determined by taking the average of  $E(p_1, \dots, p_J)$  for each of the subsets  $S_n$  separately given  $g$  as defined in (4.1). Making use of Table 1, we find for  $E(p_1, \dots, p_4)$  given the information (5.15) :

$$E(p_1, p_2, p_3, p_4) = (.46 \ .21 \ .25 \ .08) \quad (5.18)$$

We may conclude therefore, that the mean expected value method is suitable in circumstances where only an incomplete ordering of states of nature is given.

- d. When information is available of types different from the cases a. and c., it seems no longer possible to find mean expected values in an analytical way. In this case, the obvious way to proceed is the use of numerical methods, i.e. generation of a random sample of points. For example, if (3.1) would be the constraint set, the first step would be the drawing of a random sample of sufficient size from the constraint set apart from the constraints with an irregular form :

$$\left\{ \begin{array}{l} p_1 \geq p_2 \\ p_2 \geq p_3 + .1 \\ p_3 \geq p_4 \\ p_j \geq 0 \quad j = 1, 2, 3, 4 \\ \sum_j p_j = 1 \end{array} \right. \quad (5.19)$$

In the second step those elements of the random sample have to be deleted which do not satisfy the constraints from (3.1) excluded in (5.19) :

$$\left\{ \begin{array}{l} p_1 \geq p_2 + p_4 \\ p_2 \leq .4 \end{array} \right. \quad (5.20)$$

The remaining points can be used in a third step to compute the mean expected value or the mean variance of pay-offs.

## 6. Concluding Remarks

In this paper we have shown that an appropriate way to deal with ordinal information in decision problems is the use of probability distributions. If we may assume that the ordinal variables are drawn from a uniform distribution, it is possible to determine mean expected values and mean variances of alternatives in an analytical way.

Obviously, the results of the method depend on the assumption that the underlying statistical distribution is uniform. In most decision situations there will not be sufficient information available to form a basis for alternative statistical distributions. If one wants to test the sensitivity of the outcomes for the assumed distribution, this can be done by specifying alternative distributions. In that case, one has to make use of numerical methods, however.

The method developed is not only applicable in case of decision-making under uncertainty, but also in other fields of decision theory, such as multi-objective decision-making. In the latter case, the ordinal information would refer to the weights to be attached to the objectives.

## Note

- 1) The term "mean expected value" may at first sight look peculiar since in statistics, the mean of an expected value is equal to that expected value. Note however, that "expected" is used with respect to the value of the pay-off of an alternative. Thus the expected value of the pay-off of an alternative  $i$  is computed as the weighted sum of the pay-offs  $X_{ij}$ , where the appropriate weights are  $p_1, \dots, p_J$ . The expected value can be computed for any series of probabilities  $(p_1, \dots, p_J)$ . The term "mean" refers to the fact that these probabilities are drawn from a certain probability distribution.

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